臺灣大學數學系 104 學年度碩士班甄試試題

科目：線性代數

2014.10.24

Notation: $\mathbb{Q}$ is the set of rational numbers, $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers. Let n be a positive integer and I_n be the identity matrix in $M_2(\mathbb{Q})$.

Problem 1 (20 pts).

(a) For each $x \in \mathbb{R}$, let V_x be the subspace of $\mathbb{R}^4$ generated by

$$(x, 1, 1, 1), (1, x, 1, 1), (1, 1, x, 1), (1, 1, 1, x).$$

Determine all x such that $\dim_{\mathbb{R}} V_x \leq 3$.

(b) Find the dimension and a basis for the space of $\mathbb{R}$-linear maps $L : \mathbb{R}^5 \hookrightarrow \mathbb{R}^3$ whose kernels contain $(0, 2, -3, 0, 1)$.

Problem 2 (20 pts). Let

$$A = \begin{pmatrix} -1 & 4 & -2 \\ -2 & 5 & -2 \\ -1 & 2 & 0 \end{pmatrix}.$$

(1) Compute the characteristic polynomial of A.
(2) If $f(x) = (x-3)^2 + 5$, find the eigenvalues of $f(A)$.
(3) Find an orthogonal matrix $P \in M_3(\mathbb{R})$ such that $P^{-1}AP$ is diagonal.

Problem 3 (15 pts). Let $A, B \in M_n(\mathbb{R})$ be invertible matrices. Show that

(1) If $ABA^{-1}B^{-1} = c \cdot I_n$, then $c = \pm 1$;
(2) If $AB - BA = c \cdot I_n$, then $c = 0$.

Problem 4 (10pts). Let $A \in M_n(\mathbb{R})$ such that $A^3 = A$. Show that $\operatorname{rank} A = \operatorname{trace} A^2$.

Problem 5 (15pts). Let $A \in M_n(\mathbb{R})$ such that $\operatorname{rank} A + \operatorname{rank}(I_n - A) = n$. Show that $A^2 = A$.

Problem 6 (20 pts). Let $A \in M_n(\mathbb{Q})$ with $A^n = 0$ but $A^{n-1} \neq 0$. Show that if $B \in M_n(\mathbb{Q})$ commutes with A ($\Longleftrightarrow$ $BA = AB$), then

$$B = a_1 + a_2 A + \ldots + a_n A^{n-1} \text{ for some } a_1, \ldots, a_n \in \mathbb{Q}.$$