臺灣大學數學系
101 學年度碩士班甄試試題
科目：線性代數

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You should include in your answer every piece of computation and every piece of reasoning so that the corresponding partial credit could be gained.

(20%) **1.** (a) Prove that if A is a symmetric matrix then A^2 is symmetric. Is the converse true? Justify your answer.

(b) Determine all real $m \times n$ matrices A for which $A^{\mathrm{T}}A = 0$. Justify your answer.

(c) Suppose that K is a square matrix with $K = -K^{\mathrm{T}}$ and that $I - K$ is nonsingular. Let $B = (I + K)(I - K)^{-1}$. Prove that $B^{\mathrm{T}}B = BB^{\mathrm{T}} = I$.

(20%) **2.** Let V be the real vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$.

(a) For any integer n, define $f_n(x) = x + n$. Determine the dimension of the subspace of V generated by $\{f_n(x): n \in \mathbb{Z}\}$. Justify your answer.

(b) Define $g: \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \begin{cases} 4i + 1 - x, & \text{if } 4i \le x < 4i + 2 \text{ for } i \in \mathbb{Z}; \\ x - 4i - 3, & \text{if } 4i + 2 \le x < 4i + 4 \text{ for } i \in \mathbb{Z}. \end{cases}$$

For any integer n, define $g_n(x) = g(x + n)$. Determine the dimension of the subspace of V generated by $\{g_n(x): n \in \mathbb{Z}\}$. Justify your answer.

(20%) **3.** (a) Suppose I is the $n \times n$ identity matrix and J is the $n \times n$ matrix whose entries are all 1. Determine the ranks of J and $J - I$. Justify your answer.

(b) Prove that the rank of an $n \times n$ $(0,1)$-matrix A with $A_{ij} + A_{ji} = 1$ for $1 \le i < j \le n$ is either n or $n - 1$.

(20%) **4.** A square matrix is called *unimodular* if its determinant is 0 or ± 1. A matrix is called *totally unimodular* if all of its square submatrices are unimodular. It is easy to see that any entry of a totally unimodular matrix is 0 or ± 1.

(a) For any $n \ge 3$, give an $n \times n$ $(0,1)$-matrix which is not unimodular. Justify your answer.

(b) Prove that any $m \times n$ matrix in which every column has exactly one 1, exactly one -1 and all other entries 0 is totally unimodular.

(20%) **5.** (a) Prove that all eignevalues of a real symmetric matrix are real.

(b) Suppose S is an $m \times m$ real symmetric matrix whose eigenvalues are $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_m$. Recall that there is an orthonormal basis $v_1, v_2, \ldots, v_m$ for which each v_i is a corresponding eigenvector of λ_i. Prove that $\lambda_1 \ge \frac{x^{\mathrm{T}}Sx}{x^{\mathrm{T}}x} \ge \lambda_m$ for any nonzero m-vector x.

(c) Suppose A is an $n \times n$ real symmetric matrix whose eigenvalues are $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_n$. Let B be the matrix obtained from A by deleting the last row and the last column, and its eigenvalues are $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_{n-1}$. Prove that these eigenvalues are interlacing, that is $\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \cdots \ge \lambda_{n-1} \ge \mu_{n-1} \ge \lambda_n$.