臺灣大學數學系
100 學年度碩士班甄試試題
科目：線性代數 2010.10.22

(1) (20 %) Let $\mathbb{F}_p$ be the field of p elements, V be an n-dimensional vector space over $\mathbb{F}_p$, and let $1 \le m \le n$. Find the number of m-dimensional subspaces of V.

(2) (20 %) Let V be an F-vector space with basis $B = \{v_1, \ldots, v_n\}$, and V^* be the space of all linear functions $f : V \to F$. V^* has a basis $B^* = \{f_1, \ldots, f_n\}$ where $f_i(v_j) = \delta_{ij}$.

Let V and W be F-vector spaces with bases B and C, respectively. Let $T : V \to W$ be a linear transformation and A be the matrix representation of T in the bases B and C. Define the linear transformation $T^* : W^* \to V^*$ by $T^*(g)(v) = g(Tv)$ for $g \in W^*, v \in V$.

What is the matrix representation of T^* in the bases C^* and B^*?

(3) (20 %) Let A be a square matrix of dimension n over a field. Show that
(a) if rank $A = n$, then rank adj $A = n$;
(b) if rank $A = n-1$, then rank adj $A = 1$;
(c) if rank $A < n-1$, then rank adj $A = 0$.

(4) (20 %) Let the matrix

$$A = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 & a_1 \\ 0 & 0 & \cdots & \cdots & a_2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & a_{n-1} & \cdots & \cdots & 0 & 0 \\ a_n & 0 & \cdots & \cdots & 0 & 0 \end{bmatrix} \in M_n(\mathbb{C})$$

Show that A is diagonalizable if and only if for each $k = 1, 2, \ldots, n$, if $a_k = 0$, then $a_{n+1-k} = 0$.

(5) (20 %) Let $(V, \langle\ ,\ \rangle)$ be an inner product space. Let $\{v_1, \ldots v_n\}$ be an orthonormal subset of V, $x \in V$.
(a) Show that $\sum_{k=1}^{n} |\langle x, v_k\rangle|^2 \le \|x\|^2$.
(b) Let W be the subspace generated by $\{v_1, \ldots v_n\}$. Show that the following statements are equivalent:
(i) $x \in W$.
(ii) $x = \sum_{k=1}^{n} \langle x, v_k\rangle v_k$.
(iii) For any $y \in V$, $\langle x, y\rangle = \sum_{k=1}^{n} \langle x, v_k\rangle\langle v_k, y\rangle$.