臺灣大學數學系

八十七學年度碩士班甄試入學考試試題

微分方程

Choose 2 Problem from below. 15 points each.

1. Find the general solutions of the following systems
    1. $$\frac{dx}{dt} = 2x + y$$
    $$\frac{dy}{dt} = 4x + 3y$$
    2. $$\frac{dx}{dt} = 2x + y + 2\sin t$$
    $$\frac{dy}{dt} = 4x + 3y$$
2. Assume $y(x)$ satisfies
    $$\frac{dy}{dx} = \frac{y}{x}(2 - \frac{y}{x}) \ , \ y(1) = \frac{1}{2}$$
    1. Show that $\lim_{x\to\infty} \frac{dy}{dx} = 1$ and $\lim_{x\to\infty} \frac{y}{x} = 1$.
    2. Show that if $v(x)$ satisfies
    $$\frac{dv}{dx} = v(\frac{y}{x} - v) \ , \ v(1) = \frac{1}{2},$$
    then $\lim_{x\to\infty} v = 1$.
3. Consider a particle sliding on a frictionless curve C given by $x = x(s), y = y(s)$ under the influence of gravity, where s is the arc length of the curve. Let t be the time.
    1. Show that the motion of the particle satisfies the equation
    $$\frac{d^2 s}{dt^2} = -g\frac{dy}{ds},$$
    where g is the gravitational acceleration.

2. Assume $y = y_0$ when $t = 0$. Show that $\frac{1}{2}(\frac{dx}{dt})^2 = g(y_0 - y)$ and

$$t = c \pm \int [2g(y_0 - y)]^{-\frac{1}{2}} \, dx$$

for some constant c.

3. Let C be a cycloid

$$x = a(\theta + \pi + \sin\theta),$$
$$y = -a(1 + \cos\theta),$$

with parameter θ. Show that a particle oscillate on C has the period of oscillation $4\pi(\frac{a}{g})^{\frac{1}{2}}$.