臺灣大學數學系

九十九學年度碩士班甄試試題

科目：微積分與線性代數 2009.10.30

(1) (10 %) Let $\{a_n\}_{n=1}^{\infty}$ be a decreasing positive sequence. Assume that the series $\sum a_n$ diverges. Show that

$$\lim_{n\to\infty} \frac{a_1 + a_3 + \cdots + a_{2n-1}}{a_2 + a_4 + \cdots + a_{2n}} = 1.$$

(2) (20 %) Let $f(x) = (x^2 - 1)^n$.

(a) Show that, for all $1 \le k \le n-1$, the equation $f^{(k)}(x) = 0$ has roots ± 1.

(b) Show that the equation $f^{(n)}(x) = 0$ has exactly n distinct roots in the interval $(-1, 1)$.

(3) (20 %) Let $g_n(x) = \frac{d^n}{dx^n}(x^2 - 1)^n$, $n \ge 1$.

(a) Show that $\int_{-1}^{1} g_m(x) g_n(x) dx = -\int_{-1}^{1} \frac{d^{m+1}}{dx^{m+1}}(x^2-1)^{m+1} \frac{d^{n-1}}{dx^{n-1}}(x^2-1)^{n-1} dx$.

(b) Find $\int_{-1}^{1} g_n^2(x) dx$.

(c) Find $\int_{-1}^{1} g_m(x) g_n(x) dx, n \ne m$.

(4) (10 %) Let V be the real vector space spanned by $1, t, t^2, t^3, t^4, e^t, te^t, t^2e^t, t^3e^t, t^4e^t$. Let W be the subspace space spanned by $1, t, t^2, e^t, te^t, t^2e^t$. Let D be the derivative. Find the linear map $\overline{D}$ induced by D on the quotient space V/W.

(5) (20 %) Let $A = (a_{ij}(t))$ be an $n \times n$ nonsingular matrix whose entries $a_{ij}(t)$ are differential functions of real variable t. Let $A' = (\frac{d}{dt} a_{ij}(t))$. Show that $\frac{d}{dt}\det(A) = \det A \cdot \text{trace}(A' \cdot A^{-1})$.

(6) (20 %) Let A be an $n \times n$ matrix, all of whose eigenvalues are positive real numbers. Show that for any positive integer m, there exists a real matrix B such that $B^m = A$.