臺灣大學數學系

九十學年度碩士班甄試入學考試試題

高等微積分

There are problems A to E. Each problem has 25 points. You have to do at least 4 (out of 5) problems. Please write down your proof, or computation steps clearly on the answer sheets.

A.

Let $f(x,y) = e^{-\frac{1}{|x-y|}}$ for $x \neq y$, and $f(x,y) = 0$ for all $x \in R$.

(a)

Show that $f(x,y)$ is continuous in the (x,y) plane. Is $f(x,y)$ uniformly continuous in the domain $D = \{(x,y) | x - \frac{1}{x^2} < y < x + \frac{1}{x^2}\}$?

(b)

Is $f(x,y)$ differentiable in the plane? If yes, write down its total differential.

B.

Let $F(x,y)$ be a continuously differentiable function on the plane. Define

$$Z = \{(x,y) | F(x,y) = 0\}.$$

(a)

If Z contains two differentiable curves Γ_1 and Γ_2 intersecting non-tangentially (i.e., transversally) at a point P, prove that $\nabla F(P) = 0$.

(b)

If $F(x,y) = 2y^3 - 2xy^2 + (1+x^2)y - e^x$, show that Z consists of a single curve which is the graph of an infinitely differentiable function $y = \phi(x)$.

C.

Find the values of the following two integrals.

(a)

$\int_\Gamma \frac{-ydx + xdy}{x^2+y^2}$, where Γ is the ellipse $4(x+\frac{1}{3})^2 + y^2 = 1$ oriented counterclockwise.

(b)

$$\int_{\Omega} \frac{1}{\sqrt{x^2+(y-1)^2+z^2}}\, dxdydz,$$ where $\Omega = \{(x,y,z) | x^2+y^2+z^2 < 1\}$.

D.

Let $f(y) = \int_0^{\infty} \frac{\sin xy}{x(1+x^2)}\, dx$ for $y \in R$.

(a)

Find the domain of $f(y)$.

(b)

Prove that $f(y)$ is second-times continuously differentiable, and

$f''(y) - f(y) = -\frac{\pi}{2}$.

(c)

Find $f(y)$ explicitly.

E.

Determine the region of convergence of the series $\sum_{n=0}^{\infty} \frac{x^n}{1-x^n}$ for $x \in R$. Can this series be integrated, or differentiated, term by term in its region of convergence?