臺灣大學數學系

八十九學年度碩士班甄試入學考試試題

高等微積分

Choose 4 from the following 5 problems

1.

Let $f : R \to R$ be continuous on R and differentiable on $R \setminus \{0\}$. Suppose $\lim_{x \to 0} f'(x) = L \in R$. Is it true that $f'(0)$ exists and $f'(0) = L$? Prove it or give a counterexample.

2.

Determine convergence or divergence of the following improper integrals.

$$(1) \int_1^\infty \frac{1}{x^{1+\frac{1}{x}}}\,dx \quad (2) \int_1^\infty \frac{\sin x}{\sqrt{x}}\,dx \quad (3) \int_1^\infty \sqrt{x}\sin x\,dx \quad (4) \int_1^\infty \sin(x^3)\,dx.$$

3.

Assume $0 \le \alpha \le 1$. Find α such that the integral

$$\int_0^1 |xe^{x^2-1}(xe^{x^2-1} - \alpha)|\,dx$$

has an absolute minimum at α.

4.

Let M, N, Q be continuously differentiable functions of x, y, z defined on $R^3 \setminus \{(0,0,0)\}$. Find and prove a sufficient condition on M, N, Q such that you can construct a differentiable function $f(x,y,z)$ with $\frac{\partial f}{\partial x} = M, \frac{\partial f}{\partial y} = N, \frac{\partial f}{\partial z} = Q$.

5.

Let (r, θ) denote polar coordinates in R^2 and $f_k : [0, 2\pi] \to [1, 2], k = 1, 2, 3, \cdots$, be continuous functions satisfying $f_k(0) = f_k(2\pi)$. Suppose $\Omega_k \equiv \{(r,\theta) : 0 \le \theta \le 2\pi, 0 \le r \le f(\theta)\}$ is convex for each k, that is, the line segment $\overline{PQ} \subset \Omega_k$ whenever $P, Q \in \Omega_k$. Prove that there is a subsequence of $\{f_k\}$ which converges uniformly on $[0, 2\pi]$.