臺灣大學數學系

八十八學年度碩士班甄試入學考試試題

高等微積分

A.

Let $y=f(x)$ be defined on $a<x<b$

1. Use $\varepsilon-\delta$ definition to define the following terminologies:

   (1)

   f continuous on (a,b), (2) f uniformly continuous on (a,b).

2. Determine which of the following functions is uniformly continuous on $(a,b)=(0,\infty)$.

   (1)

   $\sin x^2$ (2) $\sqrt{x}$ (3) (甲組) $\sin^2 x$

3. Let (a,b) be a bounded interval, show that the following three statements are equivalent:

   (1)

   f uniformly continuous,

   (2)

   Three exists a unique continuous function g on $[a,b]$ such that

   $g(x)=f(x)$ for $a<x<b$

   (3)

   (甲組) f is continuous on (a,b), and $f(x)^2$ is uniformly continuous on (a,b).

B.

Let $f_n(x)(n=1,2,3,\cdots)$ be defined on $0\le x\le 1$.Determine whether $f_n(x)$ uniformly converges on $0\le x\le 1$ to some $f(x)$.Does

$$\lim_{n\to\infty}\int_0^1 f_n(x)dx=\int_0^1 f(x)dx\ ?$$

(1)

$f_n(x) = (n+1)x^n(1-x)$, (2) $f_n(x) = \sum_{k=0}^{n-1} \frac{1}{n} \sin^2(x + \frac{k}{n})$.

C.

Define

$$f(x,y) = \begin{cases} \frac{x^3+y^2}{x^2+y}, & when\ x^2+y \neq 0 \\ 0, & when\ x^2+y=0 \end{cases}$$

(1)

Determine the set of all points where f is continuous.

(2)

Is f uniformly continuous on the set $\{(x,y) || y| < \frac{1}{2}x^2,\ x^2+y^2<1\}$?

(3)

Find all directions along which the directional derivative of $f(x,y)$ exists at $(0,0)$.

Is $f(x,y)$ differentiable at $(0,0)$?

D.

Find the value of the following integrals,

(1)

$\int_\Gamma \frac{xdy-ydx}{x^2+y^2}$, Γ 代表曲線 $x = t\cos t$, $y = t^2 \sin t$, $2\pi \le t \le 6\pi$.

(2)

$\int_A \frac{dx}{(1+|x|^4)^{\frac{1}{4}}}$, 其中 $A = \{x = (x_1, x_2, x_3) \in^3 \ | x_2 > 0,\ x_1^2 + x_2^2 + x_3^2 \le 9\}$

E.

Let $f(x,y)$ be continuously differentiable on $D = \{(x,y) \mid x^2+y^2<1\}$.

$f(0.0) = 0$, and $|f(x,y)| \le 1$ in D. Prove that there exists a unique continuous function $z = \phi(x,y)$ defined on D such that $|\phi(x,y)| \le 1$, and $z^3 + z(x^2+y^2) = f(x,y)$ on D. Show that $\phi(x,y)$ is continuously differentiable on $D - \{(0,0)\}$. Must $\phi(x,y)$ differentiable at $(0,0)$?