臺灣大學數學系

八十六學年度碩士班甄試入學考試試題

高等微積分(甲組)

1.

(60)

a) Let $f(x,y)$ be a real-valued function defined on R^2. Give (sufficient) conditions of f_x and f_y under which f is differentiable at $\mathbf{x}_0=(x_0,y_0)\in R^2$. State it as a theorem and prove it.

b)

Check that whether the function g is differentiable at the origin, where $g(x,y)=\frac{\sin(x^4+y^4)}{x^2+y^2}$ if $(x,y)\neq(0,0)$, and $g(0,0)=0$.

c)

Let $H:R^2\to R^2$ such that $H(x,y)\equiv(x^3-2xy+y,x^2+y)$. Find the *Jacobian matrix* $H'(1,1)$ of H at (1,1). Evaluate $\|H'(1,1)\|$. Here for a linear transformation $T:R^2\to R^2$ its operator norm is defined as $\|T\|\equiv\max\{\|T(\mathbf{x})\|_2;\|\mathbf{x}\|_2=1,\mathbf{x}\in R^2\}$, where $\|\mathbf{x}\|_2=\sqrt{x^2+y^2}$ for $\mathbf{x}=(x,y)\in R^2$.

d)

State the inverse function theorem regarding the system of transformations $u=p(x,y),v=q(x,y)$. **No proof is needed.** Next consider solving the equations (same as given in c))

$$\begin{aligned} u &= x^3-2xy+y,\\ v &= x^2+y \end{aligned}$$

for (x,y) given (u,v). Show that when $|u|,|v-2|$ are small the system has a unique solution near $x=y=1$.

2.

(24) Let $\mathcal{D}_1=\{(x_1,\ldots,x_n)|x_1^2+\cdots+x_n^2\leq 1\}$ and $\mathcal{D}_2=\{(x_1,\ldots,x_n)\mid x_1^2+\cdots+x_n^2\geq 2\}$.

Find the range of $p,q\in R$ such that the following two improper integrals $I_1(p),I_2(q)$ converge:

$$I_1(p)=\int\cdots\int_{\mathcal{D}_1}\frac{dx_1\cdots dx_n}{(1-\cos\sqrt{x_1^2+\cdots+x_n^2})^p},\ \ I_2(q)=\int\cdots\int_{\mathcal{D}_2}\frac{dx_1\cdots dx_n}{(x_1^2+\cdots+x_n^2)^q\ln\sqrt{x_1^2+\cdots+x_n^2}}.$$

3.

(16) Let S be the set of continuously differentiable functions $f(x,y)$ defined on $\mathcal{D}=\{(x,y)|x^2+y^2\leq 1\}$ and put for any such function the integral

$$I(f) = \int\int_{\mathcal{D}} \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] dxdy.$$

Show that if $\varphi \in S$ is twice continuously differentiable and satisfies $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \equiv 0$ in the interior of $\mathcal{D}$, then for any $u \in S$, vanishing on the boundary of $\mathcal{D}$, one has

$$I(\varphi + u) \geq I(\varphi).$$