臺灣大學數學系

八十六學年度碩士班甄試入學考試試題

高等微積分(乙組)

1.

(60)

a) Let $f(x,y)$ be a real-valued function defined on R^2. Prove that if f_x and f_y are continuous at $\mathbf{x}_0 = (x_0, y_0) \in R^2$ then f is differentiable at $\mathbf{x}_0$.

b)

Check that whether the function g is differentiable at the origin, where $g(x,y) = \frac{\sin(x^4+y^4)}{x^2+y^2}$ if $(x,y) \neq (0,0)$, and $g(0,0) = 0$.

c)

Let $H : R^2 \to R^2$ such that $H(x,y) \equiv (x^3 - 2xy + y, x^2 + y)$. Find the *Jacobian matrix* $H'(1,1)$ of H at (1,1). Evaluate $\|H'(1,1)\|$. Here for a linear transformation $T : R^2 \to R^2$ its operator norm is defined as $\|T\| \equiv \max\{\|T(\mathbf{x})\|_2; \|\mathbf{x}\|_2 = 1, \mathbf{x} \in R^2\}$, where $\|\mathbf{x}\|_2 = \sqrt{x^2+y^2}$ for $\mathbf{x} = (x,y) \in R^2$.

d)

State the inverse function theorem regarding the system of transformations $u = p(x,y), v = q(x,y)$. **No proof is needed.** Next consider solving the equations (same as given in c))

$$\begin{aligned} u &= x^3 - 2xy + y, \\ v &= x^2 + y \end{aligned}$$

for (x,y) given (u,v). Show that when $|u|, |v-2|$ are small the system has a unique solution near $x = y = 1$.

2.

(24) Let $\mathcal{D}_1 = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \leq 1\}$ and $\mathcal{D}_2 = \{(x_1, \ldots, x_n) \mid x_1^2 + \cdots + x_n^2 \geq 2\}$.

Find the range of $p, q \in R$ such that the following two improper integrals $I_1(p), I_2(q)$ converge:

$$I_1(p) = \int \cdots \int_{\mathcal{D}_1} \frac{dx_1 \cdots dx_n}{(1 - \cos\sqrt{x_1^2 + \cdots + x_n^2})^p}, \quad I_2(q) = \int \cdots \int_{\mathcal{D}_2} \frac{dx_1 \cdots dx_n}{(x_1^2 + \cdots + x_n^2)^q \ln\sqrt{x_1^2 + \cdots + x_n^2}}.$$

3.

(16) Let S be the set of continuously differentiable functions $f(x,y)$ defined on $\mathcal{D} = \{(x,y) | x^2 + y^2 \leq 1\}$ and put for any such function the integral

$$I(f) = \int\int_{\mathcal{D}} \left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right] \, dxdy.$$

Show that if $\varphi \in S$ is twice continuously differentiable and satisfies $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} \equiv 0$ in the interior of $\mathcal{D}$, then for any $u \in S$, vanishing on the boundary of $\mathcal{D}$, one has

$$I(\varphi + u) \geq I(\varphi).$$