臺灣大學數學系

八十五學年度碩士班甄試入學考試試題

分析(應用數學組)

1.

(a)

Given a function f defined on R and three distinct points x_i, $i = 0, 1, 2$, find a polynomial P of degree 3 such that $P(x_i) = f(x_i)$ for $i = 0, 1, 2$ and $P'(x_1) = f'(x_1)$.

(b)

Derive the Simpson rule for numerical integration on interval $[a, b]$:

$$\begin{aligned}\int_a^b f(x)\,dx \;\approx\; & \tfrac{4h}{3}(f_1 + f_3 + f_5 + \cdots + f_{2m-1}) \\ & + \tfrac{2h}{3}(f_2 + f_4 + \cdots + f_{2m-2}) + \tfrac{h}{3}(f_0 + f_{2m})\end{aligned}$$

where $h = \frac{b-a}{2m}$, $f_j = f(a + jh)$.

(c)

Prove that the Simpson rule has an error bounded above by $\frac{h^4}{180}(b-a) \max_{x \in [a,b]} |f^{(4)}(x)|$.

2.

(a)

Find the rectangular parallelepiped of greatest volume inscribed in the ellipsoid: $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

(b)

Describe and prove the Lagrange's method of undetermined multiplier for the extremal problem with constrain, namely, find the extreme value of a function $f(x, y, z)$ under the constraint $g(x, y, z) = 0$.

3.

Consider damped forced motion of a mass on a spring, govern by

$$my'' + cy' + ky = A\cos(\omega t), \tag{1}$$

where m is the mass of the spring, c is a damping constant, k is the spring modulus, and A and ω are the amplitude and frequency of forcing respectively.

(a)

Write the general solution in the form of $y_h + y_p$, where y_h is the homogeneous solution by considering $A = 0$ in (1), and y_p is the particular solution. For the y_h, be sure to discuss various cases in choosing the constants m, c, and k.

(b)

What is the periodic and amplitude of the particular solution y_p?

(c)

How do we choose ω so that the amplitude of the particular solution y_p is a maximum?

4.

Solve the following linear system of ordinary differential equations:

(a)

$$\frac{du_1}{dt} = u_2, \qquad \frac{du_2}{dt} = -u_1$$

with initial condition $u_1(t=0) = u_2(t=0) = 1$.

(b)

$$\frac{du_1}{dt} = 2u_1, \qquad \frac{du_2}{dt} = -u_1 + 2u_2$$

with initial condition $u_1(t=0) = 1$ and $u_2(t=0) = 2$.

5.

The operator norm of a matrix A is defined by

$$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|},$$

where $\|x\|$ and $\|Ax\|$ are the 2-norms of x and Ax, respectively. Show that $\|A\|$ satisfies

$$\|A\|^2 = \max_j \{|\lambda_j| : \lambda_j \text{ eigenvalues of } A^T A\}.$$

(Hint: $\|A\|^2 = \max_{x \neq 0} \frac{x^T A^T A x}{x^T x}$ and $A^T A$ is symmetric and nonnegative definite.)