臺灣大學數學系
九十九學年度碩士班甄試試題
科目：高等微積分

2009.10.30

(1) (25 pts) Define $f(0,0) = 0$ and $f(x,y) = \dfrac{x^3}{3x^2+y^2}$, $(x,y) \neq (0,0)$.

(a) Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at $(0,0)$ and f is continuous at $(0,0)$.
(b) Prove that f is not differentiable at $(0,0)$.
(c) Is f^2 differential at $(0,0)$?

(2) (25 pts) Assume $a > 0$ and $b > 0$. Find all (a,b) such that $\dfrac{\sin(x^a)}{1+x^b}$ is uniformly continuous on $\{x : x > 0\}$.

(3) (25 pts)

(a) Let $\{b_n\}$ be defined by $b_1 = 1$; $b_{2m} = \dfrac{b_{2m-1}}{4}$; $b_{2m+1} = 1 + b_{2m}$. Find $\limsup_{n\to\infty} b_n$ and $\liminf_{n\to\infty} b_n$.
(b) Let $\{c_n\}$ and $\{d_n\}$ be two strictly increasing sequences of positive integers satisfying $c_n + d_n < 1.5n$. Define $p(m) = 1$ if $m \in \{c_n\}$, $p(m) = 0$ if $m \notin \{c_n\}$; $q(m) = 1$ if $m \in \{d_n\}$, $q(m) = 0$ if $m \notin \{d_n\}$, where m is a positive integer. Find $\limsup_{m\to\infty}(p(m) + q(m))$.

(4) (25 pts)

(a) Let $f(x)$ be an increasing function on $[0,1]$. Prove that f is Riemann integrable.
(b) Assume $g(x,y) < g(x,z)$ if $y < z$ and $g(x,y) < g(t,y)$ if $x < t$. Prove or disprove that the assumptions above imply $g(x,y)$ is Riemann integrable on $[0,1] \times [0,1]$.