臺灣大學數學系
九十八學年度碩士班甄試試題
科目：高等微積分 2008.10.31

(1) (20 pts) Let A be the set of all functions from the open interval $(0,1)$ to $\mathbb{R}$, and B be the set of all continuous functions from $(0,1)$ to $\mathbb{R}$.
(a) Show that there is no one-to-one correspondence between $\mathbb{R}$ and A. That is, $\mathbb{R}$ and A do not have the same cardinal number.
(b) Prove or disprove that there is a one-to-one correspondence between $\mathbb{R}$ and B.

(2) (20 pts) Let $f_n(x) = n^2 x^n \ln x$.
(a) Show that $f_n(x) \to 0$ as $n \to \infty$ for each $x \in (0,1)$.
(b) Determine whether or not f_n converges to 0 uniformly on $(0,1)$.
(c) Find $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.

(3) (20 pts) Assume that the polynomial

$$P(x) = x^3 + a_2x^2 + a_1x + a_0 = 0$$

has three different real roots for $(a_2, a_1, a_0) = (p_0, q_0, r_0)$.
(a) Prove that there is $\epsilon > 0$ such that $P(x) = 0$ has three different real roots $\lambda_1 < \lambda_2 < \lambda_3$ if $(a_2, a_1, a_0) \in V = \{(p,q,r) \in \mathbb{R}^3 : |p - p_0| + |q - q_0| + |r - r_0| < \epsilon\}$, where $\lambda_j, j = 1,2,3$ are C^1 functions of a_2, a_1, a_0.
(b) Find $\frac{\partial \lambda_j}{\partial a_i}$.

(4) (20 pts)
(a) Assume that $f(x) > 0$ is a continuous function on $[0,\infty)$ and $F(x) = \int_0^x f(t)\,dt$. Prove that

$$\int_0^\infty \frac{f(x)}{F(x)}\,dx < \infty$$

if and only if $\int_0^\infty f(x)\,dx < \infty$.
(b) Assume that $a_k > 0$ and $s_k = a_1 + a_2 + \cdots + a_k$. Prove that

$$\sum_{k=1}^{\infty} \frac{a_k}{s_k} \text{ converges}$$

if and only if $\sum_{k=1}^{\infty} a_k$ converges. (Hint: $\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+m}}{s_{N+m}} \geq 1 - \frac{s_N}{s_{N+m}}$.)

(5) (20 pts) Let W be a compact subset of $\mathbb{R}^n$ and $\{V_\alpha\}$ be an open cover of W. Prove that there is an $\epsilon > 0$ such that for each subset E of W having diameter less than ϵ, there exists a $V_{\alpha_0} \in \{V_\alpha\}$ containing E.