臺灣大學數學系

九十四學年度碩士班甄試入學試題

高等微積分

Nov, 2005

1. (15 pts) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$1 < \limsup_{n\to\infty} \sqrt[n]{a_n} < \infty.$$

   (i) Can there exist $\{c_n\}_{n=1}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} c_n = 0$ and $\sum_{n=1}^{\infty} a_n c_n < \infty$? Prove or disprove your answer. (5 pts)

   (ii) Can there exist $\{d_n\}_{n=1}^{\infty}$ a sequence of positive numbers such that $\lim_{n\to\infty} d_n = 0$ and $\sum_{n=1}^{\infty} a_n c_n = \infty$? Prove or disprove your answer. (10 pts)

2. (30 pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a positive and continuous function on $\mathbb{R}$ such that the improper Riemann integral $\int_{-\infty}^{\infty} f(x)dx$ exists. For $n \in \mathbb{N}$, let

$$h_n(x) = \begin{cases} 1 & \text{if} \quad |x| < n \\ 2 - \frac{|x|}{n} & \text{if} \quad n \le |x| \le 2n \\ 0 & \text{if} \quad |x| > 2n \end{cases},$$

   and set $g_n(x) = f(x)h_n(x)$ for $x \in \mathbb{R}$.

   (i) Can each g_n be uniformly continuous on $\mathbb{R}$? (4 pts) Can f be uniformly continuous on $\mathbb{R}$? (6 pts) Prove or disprove all your answers.

   (ii) Can $\{g_n\}_{n=1}^{\infty}$ converge to f uniformly on $\mathbb{R}$? Prove or disprove your answer. (10 pts)

   (iii) Can $\lim_{n\to\infty} \int_{-\infty}^{\infty} |g_n(x) - f(x)| = 0$? Prove or disprove your answer. (10 pts)

3. (20pts)

   (1) Calculate $\int_0^t dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_2 =?$ (4 pts)

   (2) Find the general formula to
$$\int_0^t dt_1 \cdots \int_0^{t_{n-1}} dt_n =?$$
   for all $n \in \mathbb{N}$. Prove your answer. (6pts)

   (3) Can $\limsup_{t\to+\infty} \int_0^t \cdots \int_0^{t_{n-1}} \prod_{j=1}^n (\frac{1}{1+t_j^3}) dt_1 \cdots dt_n$ exist for all $n \in \mathbb{N}$? Prove or disprove your answer. (10 pts)

4. (15 pts) Let $f : [-\pi, \pi] \to \mathbb{R}$ be a smooth and periodic function with period 2π. Suppose
$$\left| \int_{-\pi}^{\pi} f''(x)\cos(nx)dx \right| \leq 1 \quad \text{and} \quad \left| \int_{-\pi}^{\pi} f''(x)\sin(nx)dx \right| \leq 1, \forall n \in \mathbb{N},$$
where f'' denotes the second derivative of f. For $n \in \mathbb{N}$, let $a_n = \int_{-\pi}^{\pi} f(x)\cos(nx)dx$ and $b_n = \int_{-\pi}^{\pi} f(x)\sin(nx)dx$. Can the series $\sum_{n=1}^{\infty}(a_n + b_n)$ converge absolutely? (5 pts) Can the series of function $\sum_{n=1}^{\infty} n[a_n\cos(nx) + b_n\sin(nx)]$ converge uniformly on $[-\pi, \pi]$? (10 pts) Prove or disprove all your answers.

5. (20 pts) Let B denote the unit ball of $\mathbb{R}^3$ i.e. $B = \{x \in \mathbb{R}^3 : \|x\| \leq 1\}$. let $J = (J_1, J_2, J_3)$ be a smooth vector field on $\mathbb{R}$ that vanishes outside of B and satisfies divergent free i.e. $\nabla \cdot J = 0$.

   (1) Prove that $\int_B J \cdot \nabla f dx = 0$ for f a smooth, scalar-valued function defined on a neighborhood of B. (10 pts)

   (2) Find the value of $\int_B J_1 dx$ and prove your answer. (10 pts)