國立臺灣大學數學系114學年度碩士班甄試入學筆試

高等微積分

I. Let $x = (x_1, x_2, x_3)$ be coordinates of $\mathbb{R}^3$. Let

$$f(x) = \frac{x_1^3 + x_2^2 + i\ln(1 + x_2^2 + x_3^2)}{x_1^2 + x_2^2 + ix_3^2 \sin(\frac{x_1}{x_3})} \quad \text{if } x_1x_2x_3 \neq 0,$$

$$f(x) = 0 \quad \text{if } x_1x_2x_3 = 0,$$

be a function on $\mathbb{R}^3$, where $\ln$ is the natural logarithm, that is the inverse function of e^x and $i = \sqrt{-1}$. Is f a continuous function on $\mathbb{R}^3$? If not, find all $(a, b, c) \in \mathbb{R}^3$ such that $f(x)$ is not continuous at (a, b, c). (15 pts)

II. Let $x = (x_1, x_2, x_3)$ be coordinates of $\mathbb{R}^3$. Let

$$f(x) = \frac{x_1^3 + e^{-\frac{1}{x_2^2}} + i\sin^2 x_3^2}{x_1^2 + x_2^2 + ix_3^3} \quad \text{if } x_1x_2x_3 \neq 0,$$

$$f(x) = 0 \quad \text{if } x_1x_2x_3 = 0,$$

be a function on $\mathbb{R}^3$. Is f differentiable at $(0, 0, 0)$? (10 pts)

In the following, we will use the following notations: let U be an open set of $\mathbb{R}^n$. Let $\mathcal{C}^k(U)$ be the space of k-times continuously differentiable functions on U, $k \in \mathbb{N} \cup \{0\}$. Let $\mathcal{C}^\infty(U) := \cap_k \mathcal{C}^k(U)$.

III. Let $f(x, y) \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^m)$, where $x = (x_1, \ldots, x_n)$ denotes the coordinates in $\mathbb{R}^n$, $y = (y_1, \ldots, y_n)$ denotes the coordinates in $\mathbb{R}^m$. Suppose that $f(0, 0) = 0$, $\frac{\partial f}{\partial x_j}(0, 0) = 0$, $j = 1, \ldots, n$ and the matrix $\left(\frac{\partial^2 f}{\partial x_j \partial x_\ell}(0, 0)\right)_{j,\ell=1}^n$ is invertible.

(a) Show that there is an open set V of $0 \in \mathbb{R}^m$ and a smooth function $g : V \to \mathbb{R}^n$, such that $\frac{\partial f}{\partial x_j}(g(y), y) = 0$, for every $y \in V$. (10 pts)

(b) Show that there are open sets Ω, Ω_1 of $0 \in \mathbb{R}^n$ in $\mathbb{R}^n$ and a smooth function $H : \Omega_1 \to \mathbb{R}$, such that

$$\{(x_1, \ldots, x_n, \frac{\partial f}{\partial x_1}(x, 0), \ldots, \frac{\partial f}{\partial x_n}(x, 0)); x \in \Omega\}$$

$$= \{(\frac{\partial H}{\partial \xi_1}(\xi), \ldots, \frac{\partial H}{\partial \xi_n}(\xi), \xi_1, \ldots, \xi_n); \xi \in \Omega_1\}.$$

(10 pts)

IV. Let U be an open set in $\mathbb{R}^n$. Let $f_k : U \to \mathbb{R}$, $k = 1, 2, \ldots$, be functions on U. Let $f : U \to \mathbb{R}$ be a function on U.

(a) Please give a precise meaning of "f_k converges uniformly to f on U as $k \to +\infty$". (5 pts)

(b) Let

$$f_k(x) = k^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-k(|x-y|^2+|x-y|^4)+i\sum_{j=1}^n x_j^2 y_j} dy,$$

where $|x-y|^2 = \sum_{j=1}^n |x_j - y_j|^2$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Show that there is a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ such that for every compact set $K \subset \mathbb{R}^n$, f_k converges uniformly to f on K as $k \to +\infty$. Can you find $f(x)$? (15 pts)

(c) Let

$$f_k(x) = k^{\frac{n}{2}} \int_{|y| \leq M} e^{-k(|x-y|^2+|x-y|^4)+i\sum_{j=1}^n x_j^2 y_j} dy,$$

where $M > 0$ is a constant, $M < +\infty$, $|x-y|^2 = \sum_{j=1}^n |x_j - y_j|^2$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$. Show that f_k converges uniformly to a function $f(x) : \mathbb{R}^n \to \mathbb{R}$ on $\mathbb{R}^n$ as $k \to +\infty$. (15 pts)

V. Let U be an open set of $\mathbb{R}$. Let $f_k : U \to \mathbb{R}$ be smooth function on U, $k = 1, 2, \ldots$. Assume that for every compact set $K \subset U$ and every $m \in \mathbb{N} \cup \{0\}$, there is a constant $C_{K,m} > 0$ such that

$$\sup \left\{ \left| (\frac{d^m f_k}{dx^m})(x) \right| ; x \in K \right\} \leq C_{K,m},$$

for every $k = 1, 2, \ldots$. Suppose that $\lim_{k \to +\infty} f_k(x) = f(x)$, for every $x \in U$, where $f : U \to \mathbb{R}$ is a function on $\mathbb{R}$. Show that

(a) $f(x)$ is a smooth function on U, i.e. $f(x) \in \mathcal{C}^\infty(U)$. (10 pts)

(b) For every $m \in \mathbb{N} \cup \{0\}$, every compact subset $K \subset U$, $\frac{d^m f_k}{dx^m}(x)$ converges to $\frac{d^m f}{dx^m}(x)$ uniformly on K. (10 pts)