臺灣大學數學系 112 學年度碩士班甄試試題

科目：高等微積分

2022. 10. 20

1. (15 points)
Let $M = \{f : [0,\infty) \to [0,\infty); \int_0^\infty f(x)^2 dx \le 1\}$. Evaluate the following:

$$\sup_{f\in M} \int_0^\infty f(x)e^{-x}dx.$$

2. (15 points)
Assume $A \subset \mathbb{R}^n$ is compact and let $a \in A$. Suppose $\{a_n\}$ is a sequence in A such that every convergent subsequence of $\{a_n\}$ converges to a.
(1) Does the sequence $\{a_n\}$ also converge to a ? Justify your result.
(2) Now assume A is not compact and suppose $\{a_n\}$ is a sequence in A such that every convergent subsequence of $\{a_n\}$ converges to $a \in A$. Does the sequence $\{a_n\}$ also converge to a ? Justify your result.

3. (20 points)
Let M be a metric space and $A \subset M$ a compact subset. Suppose $f : A \to A$ is continuous and satisfies $d(f(x), f(y)) \ge d(x, y)$ for all x, y. Prove that f is onto A, i.e $f(A) = A$.

4. (20 points)
Define a sequence of functions $\{f_n(x)\}$ on $[0,1]$ as:

$$f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1 & \text{if } x \in (\frac{2k}{2^n}, \frac{2k+1}{2^n}], k = 0, 1, \cdots, 2^{n-1} - 1 \\ -1 & \text{if } x \in (\frac{2k+1}{2^n}, \frac{2k+2}{2^n}], k = 0, 1, \cdots, 2^{n-1} - 1 \end{cases}$$

Prove or disprove that we always have $\lim_{n\to\infty} \int_0^1 f_n(x)g(x)dx = 0$ as long as g is a continuous function.

5. (20 points)
Denote P_2 the set of all polynomials with real coefficients and degree ≤ 2. Consider the function $G : P_2 \to \mathbb{R}$ by

$$G(p) = \int_0^1 p(x)^2 dx.$$

Let $S = \{p \in P_2; p(1) = 1\}$. Does G attain any extremal value on S ? If yes, find $p \in S$ such that G attains an extremal value at p.

6. (10 points)
Suppose $f(x) : [a,b] \to \mathbb{R}$ is Riemann integrable, and $g(x) : \mathbb{R} \to \mathbb{R}$ satisfies $|g(x) - g(y)| \le C|x - y|$ for all x, y. Prove that $g(f(x))$ is Riemann integrable.