臺灣大學數學系 109 學年度碩士班甄試試題
科目：高等微積分

2019. 10. 18

1. (15 points)
Consider a series $\sum_{k=2}^{\infty} a_k \sin(kx)$, where $\{a_k\}$ is a sequence of real numbers. Is it possible to construct a sequence $\{a_k\}$ such that $\sum_{k=2}^{\infty} a_k \sin(kx)$ converges uniformly to $\sin(x)$ on $[0, \pi]$? Justify your result.

2. (15 points)
Suppose $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous. Show that f must be bounded by a linear function, i.e, there exist constants A, B such that $|f(x)| \le A + B|x|$ for all x.

3. (20 points)
(1): Let $f : M \to N$ be a map from metric space M to another metric space N with the property that if a sequence $\{p_n\}$ in M converges, then the sequence $\{f(p_n)\}$ in N also converges. Is f continuous ? Justify your result.
(2): Let $f : M \to \mathbb{R}$ where M is a metric space, and define $G = \{(x, y) \in M \times \mathbb{R}; y = f(x)\}$. If G is compact (in product space $M \times \mathbb{R}$). Is f continuous ? Justify your result.

4. (20 points)
Suppose $L : \mathbb{R}^3 \to \mathbb{R}^3$ is an invertible linear map and $g : \mathbb{R}^3 \to \mathbb{R}^3$ has continuous first order partial derivatives and satisfies $||g(x)|| \le C||x||^{1+\epsilon}$ for some $\epsilon > 0$ and all $x \in \mathbb{R}^3$ ($||\cdot||$ is the usual Euclidean norm). Show that $f(x) = L(x) + g(x)$ is invertible near the origin 0.

5. (20 points)
(1): Find the volume of the ellipsoid $(x + 2y)^2 + (x - 2y + z)^2 + 3z^2 \le 1$.
(2): Let C be a positively oriented simple closed curve. Find the curve C that maximizes the integral $\int_C y^3 dx + (3x - x^3) dy$.

6. (10 points)
Let $f(x)$ be a real valued continuous function on $[0, 1]$. Find the limit

$$\lim_{n \to \infty} (n + 1) \int_0^1 x^n f(x) dx.$$