臺灣大學數學系 108 學年度碩士班甄試試題
科目：高等微積分

2018.10.19

(1) [10 分] Does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n \log(n+1)}{n}$$

converge? Does it converge absolutely? Justify your answer.

(2) [10+10 分] Consider the function

$$f(x,y) = \frac{1}{(1-xy)^2}$$

defined on $\Omega = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 1\ ,\ 0 \le y \le 1\ ,\ (x,y) \neq (1,1)\}$.

(a) For any $\kappa \in (0,1)$, let $U_\kappa = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le \kappa\ ,\ 0 \le y \le \kappa\}$. Is $f(x,y)$ uniformly continuous on U_κ? Justify your answer.

(b) Is $f(x,y)$ uniformly continuous on Ω? Justify your answer.

(3) [10+15 分] For any $n \in \mathbb{N}$, consider $f_n(x) = n\, x^n\, (1-x)$ on $I = \{x \in \mathbb{R} \mid 0 \le x \le 1\}$.

(a) Determine $\lim_{n\to\infty} f_n(x)$ for every $x \in I$.

(b) Is the convergence uniform on I? Give your reason.

(4) [15+10 分] Let

$$F(x) = \int_0^\infty \frac{1-\cos(xt)}{t^2 e^t}\, \mathrm{d}t\ .$$

(a) Can you switch the order of integration and differentiation to obtain the formulae for $F'(x)$ and $F''(x)$? Explain the reason.

(b) Find the explicit[1] formula for $F'(x)$ and $F(x)$.

(5) [10+10 分] Consider

$$\begin{array}{rccl} F: & \mathbb{R}^4 & \to & \mathbb{R}^2 \\ & (x,y,u,v) & \mapsto & \left(\displaystyle\int_{x-y^2}^{x^2+y} (e^{t^2}+u)\mathrm{d}t, x^3+v \right) \end{array}.$$

(a) Prove that near $(1,1,0,0)$, the two equations $F(x,y,u,v) = (\int_0^2 e^{t^2}\mathrm{d}t, 1)$ can be solved for u,v as continuously differentiable functions of x,y.

(b) For the functions $u(x,y)$ and $v(x,y)$ in part (a), find all their first order partial derivatives at $(x,y) = (1,1)$.

[1] Not an improper integral.