臺灣大學數學系107學年度碩士班甄試試題
科目：高等微積分

2017.10.21

1. (15 points. **No partial credit will be given if the answer is wrong.**) Evaluate the integral

$$\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \frac{1}{1+3\sin^2 x}dx. \quad (\textbf{Hint.}\text{ Let } u=\tan x.)$$

2. (15 points) State and prove Leibniz's criterion for convergence of alternating series.

3. (15 points) Let F be an $\mathbf{R}$-valued C^∞ function on $\mathbf{R}^2$ such that at a point $p=(a,b)$ we have

$$\frac{\partial F}{\partial x}(p)=\frac{\partial F}{\partial y}(p)=0,\ \frac{\partial^2 F}{\partial x^2}(p)>0, \text{ and } \frac{\partial^2 F}{\partial x^2}(p)\frac{\partial^2 F}{\partial y^2}(p)-\left(\frac{\partial^2 F}{\partial x\partial y}(p)\right)^2>0.$$

Show that there exists $R>0$ such that $F(p)<F(q)$ for $q\in\{(x,y)\in\mathbf{R}^2\,|\,(x-a)^2+(y-b)^2<R^2\}$.

4. We adopt the following definitions.

Let (X,d) be a metric space. (i) A family $\mathcal{F}$ of $\mathbf{R}$-valued functions on X is *equicontinuous at a point* $x_0\in X$ (with respect to d) if

$$\forall\,\varepsilon>0 \quad \exists\,\delta>0 \quad \forall\, f\in\mathcal{F} \quad \forall x\in X \quad d(x.x_0)<\delta \ \Rightarrow\ |f(x)-f(x_0)|<\varepsilon.$$

(ii) A sequence f_n of $\mathbf{R}$-valued functions on X *converges compactly* to some function f on X if for every compact subset K of the metric space (X,d) the sequence $f_n|_K$ converges to $f|_K$ uniformly.

(15 points) Let f_n be a sequence of $\mathbf{R}$-valued functions on a metric space (X,d) which converges *pointwise* to a *continuous* function f on (X,d). Suppose that $\{f_n\,|\,n\in\mathbf{N}\}$ is equicontinuous at every point of X. Show that f_n converges compactly to f on X.

5. (15 points.) Compute the <u>outward</u> flux of the vector field $(x+ye^z,\ e^x\sin(yz),\ ye^{zx})$ through the boundary of the region $D=\left\{(x,y,z)\in\mathbf{R}^3 \,\middle|\, \left(\sqrt{x^2+y^2}-3\right)^2+z^2<1\right\}$.

6. (25 points) Show that the function $f(x)=\displaystyle\sum_{n=0}^{\infty}\frac{\cos(ne)}{n^x}$ (where $e=\sum_{m=0}^{\infty}\frac{1}{m!}$) is well-defined (i. e., the series converges) on $(0,\infty)$ and is continuous.

7. (30 points. In your argument if any theorems are used you have to clearly verify that their conditions are fulfilled.) Let f and g be $\mathbf{R}$-valued C^∞ functions on $\mathbf{R}^2$ and let $S=\{(x,y)\in\mathbf{R}^2\,|\,f(x,y)=0\}$. Suppose that at some point $p=(a,b)\in S$ we have $\frac{\partial f}{\partial x}(p)=-1,\ \frac{\partial f}{\partial y}(p)=2,\ \frac{\partial g}{\partial x}(p)=3,\ \frac{\partial g}{\partial y}(p)=-6,$

$$\begin{bmatrix}\frac{\partial^2 f}{\partial x^2}(p) & \frac{\partial^2 f}{\partial x\partial y}(p)\\ \frac{\partial^2 f}{\partial y\partial x}(p) & \frac{\partial^2 f}{\partial y^2}(p)\end{bmatrix}=\begin{bmatrix}1 & 3\\ 3 & 0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}\frac{\partial^2 g}{\partial x^2}(p) & \frac{\partial^2 g}{\partial x\partial y}(p)\\ \frac{\partial^2 g}{\partial y\partial x}(p) & \frac{\partial^2 g}{\partial y^2}(p)\end{bmatrix}=\begin{bmatrix}3 & -1\\ -1 & 2\end{bmatrix}.$$

Show that there exists $R>0$ such that $g(p)<g(q)$ for $q\in S\cap\{(x,y)\in\mathbf{R}^2\,|\,(x-a)^2+(y-b)^2<R^2\}$.