臺灣大學數學系 106 學年度碩士班甄試試題

科目：高等微積分

2016. 10. 21

1. (15 points) Let A be the unit ball $B_1(0)$ in $\mathbf{R}^3$. Compute

$$\int_A \cos(x+y+z)\,dx\,dy\,dz.$$

2. Let f be a real-valued function on $\mathbf{R}$ which has period 2π and is Riemann integrable on $[-\pi,\pi]$. We define its Fourier coefficients

$$a_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)dx\ (n=0,1,2,\ldots) \text{ and } b_n := \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx\ (n=1,2,\ldots).$$

(1) (10 points) Show that $f(x)^2$ is Riemann integrable on $[-\pi,\pi]$.

(2) (15 points) Show that the series $\frac{1}{2}a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)$ converges.

3. (1) (15 points) Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers and let $B_n = b_1 + \cdots + b_n$ ($n \in \mathbf{N}$). Suppose that $a_n \searrow 0$ as $n \to \infty$ and that there exists $M > 0$ such that $|B_n| \leqslant M$ for every $n \in \mathbf{N}$. Show that the series $\sum_{n=1}^{\infty} a_n b_n$ converges.

(2) (10 points) Show that the function series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\sin(nx)$$

converges uniformly on $[-K,K]$ if $|K| < \pi$.

4. Definition. Let $\mathcal{F}$ be a set of real-valued functions on a set X. $\mathcal{F}$ is *uniformly bounded* if there exists $M > 0$ such that $|f(x)| \leqslant M$ for every $x \in X$ and every $f \in \mathcal{F}$.

(20 points) Let $\{F_n\}_{n=1}^{\infty}$ be a sequence of convex functions on $[-2,2]$ and let $f_n = F_n|_{[-1,1]}$ ($n \in \mathbf{N}$). Suppose that $\{F_n \mid n \in \mathbf{N}\}$ is uniformly bounded. Show that there exists a subsequence of $\{f_n\}_{n=1}^{\infty}$ which is uniformly convergent on $[-1,1]$.

5. (15 points) Let $f = (f_1,\ldots,f_n) : U \to \mathbf{R}^n$ be a C^1 map from an open set U in $\mathbf{R}^n$, and let $g : V \to U$ be a continuous map from an open set V in $\mathbf{R}^n$. Suppose that

$$\det\left(\frac{\partial f_j}{\partial x_k}(x)\right) \neq 0 \text{ for every } x \in U,$$

and that $f(g(x)) = x$ for every $x \in V$. Show that g is C^1.