臺灣大學數學系 105 學年度碩士班甄試試題

科目：高等微積分

2015. 10. 22

1. 30% (each 10%)

(i) Calculate $\sup_{x\neq 0}\dfrac{x^2-\ln\left(1+x^2\right)}{x^4}=?$

(ii) Calculate $\iint_A xy\sin\left(x^2-y^2\right)dxdy$, where

$A=\left\{(x,y):0<y<1, x>y \text{ and } x^2-y^2<1\right\}$

(iii) Prove that $\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}=\dfrac{\pi^2}{8}$ (Hint: use Fourier series)

2. 30% (each 10%)

Let $C=\left\{\sum_{n=1}^{\infty}\dfrac{a_n}{3^n}:a_n\in\{0,2\}, n\in\mathbb{N}\right\}$.

(i) Prove that $C\subset[0,1]$.

(ii) Does C have interior point? Justify your answer.

(iii) Is C a compact subset of $\mathbb{R}$? Justify your answer.

3. 20% (each 10%)

Let $A=\left\{f\in C^0[0,1]:\|f\|_{C^0}\leq 1\right\}$ and $B=\left\{f\in C^1[0,1]:\|f\|_{C^1}\leq 1\right\}$,

where $C^0[0,1]=\left\{f:[0,1]\to\mathbb{R}\mid f \text{ is continuous on } [0,1]\right\}$,

$C^1[0,1]=\left\{f:[0,1]\to\mathbb{R}\mid f \text{ is of } C^1 \text{ on } [0,1]\right\}$, $\|\cdot\|_{C^0}$ is the standard sup-norm, and $\|\cdot\|_{C^1}$ is the standard C^1-norm.

(i) Is A (sequentially) compact? Justify your answer.

(ii) Prove that any sequence of B has a convergent subsequence in A.

4. 20%

Let $f:[0,1]\to\mathbb{R}$ be a continuous function. Let $F(x)=\int_0^x f(t)\,dt$ for $x\in[0,1]$. Suppose $F(1)=0$. Determine if the value

$$\sup_{g\in C^0[0,1]} \frac{\int_0^1 F(x)g(x)\,dx}{\left(\int_0^1 G^2(x)\,dx\right)^{1/2}}$$

exists, where $G(x)=\int_0^x g(t)\,dt$ for $x\in[0,1]$, and $C^0[0,1]=\{g:[0,1]\to\mathbb{R} \mid g \text{ is continuous on } [0,1]\}$.

Justify your answer.