臺灣大學數學系 102 學年度碩士班甄試試題
科目：高等微積分　　2012.10.19

Advanced Calculus

1. (40 points) True or False. Prove or disprove each of the following statements.

   a) If the real series $\sum_{j=1}^{\infty} b_j = 1$ is conditional convergent (which is not absolutely convergent), then there exist a rearrangement of $\{b_j\}$ such that $\sum_{j=1}^{\infty} b_{\sigma(j)} = 2012$, where $\{\sigma(j)\}_{j=1}^{\infty}$ is a permutation of $\{j\}_{j=1}^{\infty}$.

   b) If, for some positive number $M > 0$, the partial sum $|\sum_{j=1}^{n} b_j| \leq M$ for all $n \in \mathbb{N}$ and $\{a_j\}_{j=1}^{\infty}$ is a positive decreasing sequence that tends to 0, then $\sum_{j=1}^{\infty} a_j b_j$ converges.

   c) If $f(x)$ is a continuous function defined on the closed interval $[-1, 1]$ such that $f'(0) = 1$, then there exists a $\delta > 0$ such that $f(x)$ is an increasing function for all $x \in [-\delta, \delta]$.

   d) Suppose $g(x)$ is a nonnegative continuous function defined on the closed interval $[0, 1]$ and $f(x)$ is a positive monotone continuous function, then there exists a number $\xi \in [0, 1]$ such that

$$\int_0^1 f(t)g(t)dt = f(\xi)\int_0^1 g(t)dt.$$

2. (50 points) Evaluate

$$\int_0^1 x^9(10\ln x + 1)dx, \qquad \int_2^4 \int_{4/x}^{(20-4x)/(8-x)} (y-4)dydx,$$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2 dydx, \qquad \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i},$$

$$\int_A xdxdydz,$$

where A is the region in $\mathbb{R}^3$ bounded by the planes $x = 0$, $y = 0$, $z = 2$ and the surface $z = x^2 + y^2$.

3. (10 points) Find the maximum of $f(x, y, z) = x + y + 2z$ under the constraints

$$x^2 - xy + y^2 + z^2 = 2.$$

At which point does $f(x, y, z)$ achieve its maximum ?