臺灣大學數學系
101 學年度碩士班甄試試題
科目：高等微積分　　2011.10.21

1. (30 points) Calculate

(1)

$$\int\int_{\mathbb{R}^2} e^{-(13x^2+16xy+5y^2)}dxdy$$

(2)

$$\int_0^\infty \frac{\sin x}{x}dx.$$

2. (30 points)

(1) Suppose n is a positive integer, $a_i \geq 0$, $b_i \geq 0$ for $i = 1, 2, \cdots, n$, and p and q are two positive numbers such that $1/p + 1/q = 1$. Prove the following Holder's inequality.

$$\sum_{i=1}^{n} a_i b_i \leq (\sum_{i=1}^{n} a_i^p)^{1/p}(\sum_{i=1}^{n} b_i^q)^{1/q}.$$

(2) Suppose $a_1 \geq a_2 \geq a_3 \geq a_4$ and $b_1 \geq b_2 \geq b_3 \geq b_4$. Prove the following inequality.

$$a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4 \geq a_1b_2 + a_2b_4 + a_3b_1 + a_4b_3.$$

3. (30 points) Prove or disprove the following statements.

(1) Let $f(x, y)$ be a real-valued function on $\mathbb{R}^2$ such that both $\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y}$ and $\frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}$ exist. Then $\frac{\partial^2 f(x_0,y_0)}{\partial x \partial y} = \frac{\partial^2 f(x_0,y_0)}{\partial y \partial x}$.

(2) Let $f_n(x)$ be a sequence of continuous functions defined on $[0, 1]$. If $f_n(x)$ converges uniformly on $[0, 1]$ as $n \to \infty$, then $\lim_{n\to\infty} f_n(x)$ is a continuous function on $[0, 1]$.

4. (10 points) Suppose $f(x)$ is a continuous function defined on $[0, 1]$. Prove that

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0.$$