臺灣大學數學系
100 學年度碩士班甄試試題
科目：高等微積分

2010.10.22

(1) (25 pts) Suppose the series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$. Show that f is continuous and differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \text{ for } |x| < R.$$

(2) (25 pts) Let $\{f_n\}$ and f be defined on $[0,2)$. Suppose that $\{f_n\}$ are continuous and

$$\lim_{n\to\infty} f_n(x_n) = f(x)$$

for every sequence $\{x_n\} \subset [0,2)$ such that $\lim_{n\to\infty} x_n = x$ and $x \in [0,2)$.
(a) Is it true that $\{f_n\}$ converges uniformly to f on $[0,2)$?
(b) Is it true that f is continuous on $[0,2)$?

(3) (25 pts)
(a) Prove that

$$|\int_0^1 f(x)g(x)\,dx| \le \left(\int_0^1 f^2(x)\,dx\right)^{\frac{1}{2}} \left(\int_0^1 g^2(x)\,dx\right)^{\frac{1}{2}}.$$

(b) Let $h(x)$ be a continuous function on $[0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 h(x)\sin(nx)\,dx = 0.$$

(4) (25 pts) Let $\mathbb{N} = \{1,2,3,...\}$ denote the natural numbers and E be defined as follows: $A \in E$ if and only if A is a subset of $\mathbb{N}$. Show that there is a one-to-one and onto mapping from E to the open interval $(0,1)$.