台灣大學數學系

九十二學年度第二學期博士班資格考試題

高等統計推論

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1.

Assume that a family of probability density functions $\{f_T(t|\theta) : \theta \in \Theta\}$ has an monotone likelihood ratio in T, i.e., $h(t) = f_T(t|\theta_2)/f_T(t|\theta_2)$ is a non-decreasing function of t for any $\theta_2 > \theta_1$ and t in $\{t : f_T(t|\theta_1) > 0 \ and \ f_T(t|\theta_2) > 0\}$.

(1a)

(8%) Let $\varphi(T) = 1_{\{T>t\}} + \gamma 1_{\{T=t\}}$ with $0 < \gamma < 1$. Show that $E[\varphi(T)|\theta_2] \geq E[\varphi(T)|\theta_1]$ for any $\theta_2 > \theta_1$.

(1b)

(7%) Let X and U be independent random variables from $N(\mu, 1)$ and χ_p^2, respectively. Here, μ is an unknown non-zero parameter and p is a known positive integer. Show that the probability density functions of $T^* = X/\sqrt{\chi_p^2/p}$ has an monotone likelihood ratio in T^*.

2.

Let $X_1, \cdots, X_n$ be a random sample from a probability density function $f(x|\theta)$ with $\theta \in \Theta$. Assume that $S = S(X_1, \cdots, X_n)$ is a sufficient statistic for θ and the family of probability density functions $\{f_S(s|\theta) : \theta \in \Theta\}$ has an monotone likelihood ratio in S.

(2a)

(10%) Consider the hypotheses $H_0 : \theta = \theta_1 \ or \ \theta = \theta_2$ versus $H_A : \theta = \theta_3$, where $\theta_1 < \theta_3 < \theta_2$. Show that a test Υ with test function

$\varphi(S) = 1_{\{s_1 < S < s_2\}} + \gamma_1 1_{\{S=s_1\}} + \gamma_2 1_{\{S=s_2\}}$ satisfying

$E[\varphi(S)|\theta_1] = E[\varphi(S)|\theta_2] = \alpha$ is a most powerful level α test, $0 < \alpha < 1$.

(2b)

(5%) Show that Υ is an unbiased test.

(2c)

(8%) Show that a uniformly most powerful level α test does not exist for hypotheses

$H_0: \theta = \theta_0$ versus $H_A: \theta \neq \theta_0$.

3.

Let $X = (X_1^T, X_2^T)^T \sim N_p(\mu, \Sigma)$, where X_1 and X_2 are separately $q \times 1$ and $(p-q) \times 1$ random vectors, and Σ is at least positive semi-definite.

(3a)

(8%) Show that the moment generating function of X is $M_X(t) = \exp(t^T\mu + t^T\Sigma t)$. where t is in the neighborhood of the zero vector.

(3b)

(7%) Assume that Σ is positive definite. Derive the probability density function of X and the conditional probability density function of X_2 given $X_1 = x_1$.

4.

Let $L(\theta|\mathbf{X}, \mathbf{Y})$ and $L(\theta|\mathbf{Y})$ denote separately the likelihood functions of $(\mathbf{X}, \mathbf{Y})$ and $\mathbf{Y}$, where $\mathbf{X} = (X_1, \cdots, X_m)$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$ are the missing data and the observable incomplete-data.

(4a)

(5%) State the E-M algorithm for the estimation of θ.

(4b)

(10%) Let $\{\widehat{\theta}_r\}$ be the sequence of estimators of θ from E-M algorithm. Show that

$$L(\widehat{\theta}_{(r+1)}|\mathbf{Y}) \geq L(\widehat{\theta}_{(r)}|\mathbf{Y}).$$

5.

Let $X_1, \cdots, X_n$ be a random sample from a full-rank exponential distribution with probability density function $f(x|\theta) = \exp(\sum_{j=1}^k \theta_j W_j(x) + C(x) + D(\theta))$, where $\theta = (\theta_1, \cdots, \theta_k)^T$. Moreover, assume that $D(\theta)$ is twice differentiable with respect to θ and $-\frac{\partial^2 D(\theta)}{\partial\theta\partial\theta^T}$ is positive definite.

(5a)

(7%) Show that the maximum likelihood estimtors, say, $\widehat{\theta}_n$ of θ can be obtained via solving the equations $\sum_{i=1}^n W_j(X_i) = E[\sum_{i=1}^n W_j(X_i)|\theta]$, $j = 1, \cdots, k$.

(5b)

(8%) Show that the maximum likelihood estimtors $\widehat{\theta}_n$ are the functions of the minimum sufficient statistics.

(5c)

(5%) Let $\tau(\theta)$ be the differentiable function of θ. State the assumptions for the asymptotic efficiency of $\tau(\widehat{\theta}_n)$, and derive the asymptotic variance of $\tau(\widehat{\theta}_n)$.