臺灣大學數學系

九十一學年度第一學期碩博士班資格考試題

統計(Statistics)

Sept 11, 2002

- (25 points) A random sample is drawn from a population that is normally distributed with mean θ and variance θ, where $\theta > 0$.

  (11) (10 points) Show that the MLE of θ, $\hat{\theta}$, is a root of the quadratic equation $\theta^2 + \theta - W = 0$, where $W = n^{-1}\sum_{i=1}^{n} X_i^2$, and determine which root equals the MLE.

  (12) (15 points) Find the approximate variance of $\hat{\theta}$.
- (20 points) For a random sample $X_1, \ldots, X_n$ of Bernoulli(p) variables, it is desired to test

  $$H_0 : p = 0.49 \quad \text{versus} \quad H_1 : p = 0.51.$$

  The test being considered is to reject H_0 if $\sum_{i=1}^{n} X_i$ is large.

  Note that X_1 will take on the value of 0 or 1 with probability $1-p$ and p, respectively.
- (20 points) Suppose that $U_1, U_2, \ldots, U_m$ are iid uniform(0,1) random variables, and let $S_n = \sum_{i=1}^{n} U_i$. Define the random variable N by

  $$N = \min\{k : S_k > 1\}.$$

  (31) (5 points) Show that $P(S_k \le t) = t^k/k!$.

  (32) (10 points) Show that $E(N) = e$.

  (33) (5 points) How large should n be so that you are 95% confident that you have the first four digits of e correct?
- (20 points) X and Y are independent random variables with $X \sim exponential(\lambda)$ and $Y \sim exponential(\mu)$. (The density function of X is $\lambda^{-1}\exp(-x/\lambda)$ where $x > 0$.) It is impossible to obtain direct observations of X and Y. Instead, we observe the random variables Z and W, where

$$Z = \min\{X, Y\} \quad \text{and} \quad W = \begin{cases} 1 & \text{if } Z = Z \\ 0 & \text{if } Z = Y. \end{cases}$$

(41) Find the joint distribution of Z and W.

(42) Prove that Z and W are independent.

- (15 points) Find a $(1-\alpha)$ confidence interval for θ, given $X_1, \ldots, X_n$ iid with pdf $f(x|\theta) = 2x/\theta^2$, where $0 < x < \theta$ and $\theta > 0$.