臺灣大學數學系

八十八學年度第二學期碩博士班資格考試試題

統計與機率

機率(機率組)

1. (25/105) For any random variable X on $(\Omega, \mathcal{F}, P)$ denote the *characteristic function* of X by $\varphi_X(t) = E[e^{itX}]$.

   (1.1)
   Show that $\varphi_X(t)$ is uniformly continuous for $t \in \mathbb{R}$.

   (1.2)
   Let X be a *Normal* random variable with mean μ and variance σ^2. Find $\varphi_X(t)$.

   (1.3)
   Apply (1.2) to evaluate $E[Y^n], n \in \mathbb{N}$, where Y is a Normal random variable with mean 0 and variance σ^2. State clearly which properties or facts you are using to solve this question.

2. (30/105) Let $(\Omega, \mathcal{F}, P)$ be a fixed probability space.

   (2.1)
   Let $A_n \in \mathcal{F}, n \in \mathbb{N}$. Prove *Borel-Cantelli lemma*: If $\sum_{n=1}^{\infty} P(A_n) < \infty$, then $P(A_n \ i.o.) = 0$.

   (2.2)
   Let (X_n) be a sequence of random variables. Apply (2.1) to prove that as $n \to \infty, X_n$ converges to X in probability, denoted as $X_n \xrightarrow{P} X$, if and only if each subsequence (X_{n_k}) contains a further subsequence $(X_{n_{k(j)}})$ which converges to X almost surely.

   (2.3)
   Let $(Y_n), (Z_n)$ be two sequences of random variables such that $Y_n \xrightarrow{P} Y, Z_n \xrightarrow{P} Z$. Show that $Y_n + Z_n \xrightarrow{P} Y + Z$ and $Y_n Z_n \xrightarrow{P} YZ$.

   (Note that if you cannot prove (2.2), you can still apply it to establish (2.3).)

3. (20/105) Let P_N be the product of N Bernoulli measures of $(\eta_1, \cdots, \eta_N)$ on $\Omega_N = \{0,1\}^N$ such that

$P(\eta_k = 1) = p, P(\eta_k = 0) = 1 - p, 1 \le k \le N, 0 < p < 1$. Put

$S_N = \eta_1 + \cdots + \eta_N$.

(3.1)

Denote by $P_{N,m}$ the conditional distribution of P_N given

$S_N = m, m \in \{0, 1, ..., N\}$. Find $P_{N,m}$.

(3.2)

Evaluate $\lim_{N\to\infty} N^{-1} \ln(P_{N,S_N}(\eta))$ by using the limiting behavior of S_N/N.

機率(統計組)

1. (15/105) (1.1) Let X be a *Normal* random variable with mean μ and variance σ^2. Find its *moment generating function* $\varphi_X(t) = E[e^{tX}], t \in \mathbb{R}$.

   (1.2) Apply (1.1) to evaluate $E[Y^n], n \in \mathbb{N}$, where Y is a Normal random variable with mean 0 and variance σ^2.

2. (15/105) Let P_N be the product of N Bernoulli measures of $(\eta_1, \cdots, \eta_N)$ on

   $\Omega_N = \{0,1\}^N$ such that

   $P(\eta_k = 1) = p, P(\eta_k = 0) = 1 - p, 1 \le k \le N, 0 < p < 1$. Put

   $S_N = \eta_1 + \cdots + \eta_N$.

   (2.1) Denote by $P_{N,m}$ the conditional distribution of P_N given

   $S_N = m, m \in \{0, 1, ..., N\}$. Find $P_{N,m}$.

   (2.2) Evaluate $E[\eta_1\eta_2 \mid S_N = m]$.

統計(機率組做 **1,2** 題**,** 統計組全做)

1. (15 points) Assume $X_1, \ldots, X_n$ are i.i.d. according to $U(0, \theta)$, $\theta > 0$.

   (i)

   (5 points) Find the maximum likelihood estimator $\hat{\delta}_n$ of $(\theta - 1)^2$.

   (ii)

   (10 points) It is known that the limit distribution of $n(\theta - X_{(n)})$ is exponential distributed with parameter θ (i.e., The density function $f(y) = \theta^{-1}\exp(-y/\theta)$.). Here $X_{(n)}$ is the largest order statistic. Use this result to determine the nondegenerate limit distribution of $\hat{\delta}_n - (\theta - 1)^2$ under proper normalization.

2. (15 points) Let $X_1, \dots, X_m$ and $Y_1, \dots, Y_n$ be independent normal $N(\xi, \sigma^2)$ and $N(\eta, \tau^2)$, respectively, and consider the test of $H: \sigma^2 = \tau^2$ against $\sigma^2 < \tau^2$ with rejection region

$$\sqrt{(m+n)\rho(1-\rho)/2}[\log S_Y^2 - \log S_X^2] \geq z_\alpha,$$

where $\rho = \lim_{n\to\infty} m/(m+n)$, $0 < \rho < 1$, $S_X^2 = \sum(X_i - \bar{X})^2/(m-1)$, $S_Y^2 = \sum(Y_j - \bar{Y})^2/(n-1)$, and $P(Z > z_\alpha) = \alpha$ where Z is a standard normal random variable. Show that this test has asymptotic level α.

3. (10 points) The random variable Y has a binomial distribution with an unknown number θ of trials, and known probability of success, $1/2$. (Namely, $Y \sim Bin(\theta, 1/2)$.) Find an approximated 95% confidence interval of θ of the form $[0, c]$ where c is to be determined.

4. (15 points) Suppose that the independent pairs of random variables $(Y_1, Z_1), \dots, (Y_n, Z_n)$ are such that Y_j and Z_j are independent in $N(\xi_j, \sigma^2)$ and $N(\beta\xi_j, \tau^2)$, respectively.

   (i) (8 points) Use the method of moment to derive the estimate β.
   (ii) (7 points) Is the estimate obtained in (i) consistent?

5. (15 points) In life-testing experiments, it is quite often that the experiment is terminated whenever the first r failures have occurred among n tested units. Suppose the survival time of a particular units follows an exponential distribution $Exp(\theta)$. (i.e., $P(X > x) = \exp(-\theta x)$.)

   (i) Derive the maximum likelihood estimate of θ under the above setting.
   (ii) Discuss whether the resulting estimator is consistent when $r = 2$.