臺灣大學數學系

八十八學年度第一學期碩博士班資格考試試題

統計與機率

*機率

1. (i) Let (Ω, F, P) be a probability space and $B = \{A \in F | P(A) = 0 \ or \ 1\}$. Prove that B is a sub-σ-field of F.
   (ii) Let (X_n) be a sequence of independent random variables defined on (Ω, F, P) and A be a tail event of (X_n). Show that $P(A) = 0$ or 1.
2. Let (X_n) be a sequence of random variables converges to X almost surely. Show that (X_n) converges to X in probability. The converse is true or false?
3. Let (X_n) be a sequence of indepent random variables. Show that $\sum_{n=1}^{\infty} X_n$ converges almost surely if and only if $\sum_{n=1}^{\infty} X_n$ converges in probability.
4. Let (X_n) be a sequence of i.i.d. random variables with $E(X_n) = \mu$ and $Var(X_n) = \sigma^2$, $0 < \sigma^2 < \infty$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Show that $(S_n - n\mu)/\sqrt{n}\sigma$ converges in distribution to $N(0,1)$.

*統計

5. Let $X_1, X_2, \cdots, X_n$ be a random sample from $p.d.f.$

$$f(x;\theta) = \begin{cases} \frac{1}{\theta} & , \quad 0 < x \le \theta, 0 < \theta < \infty \\ 0 & , \quad \text{elsewhere} \end{cases}$$

   (i) Find the maximum likelihood estimator $\hat{\theta}$ for the parameter θ.
   (ii) Show that $\hat{\theta}$ is a biased estimator.
6. Let $X_1, X_2, \cdots, X_n$ denote a random sample from the distribution that has $p.d.f.$

$$f(x;\theta) = \frac{1}{\sqrt{2\pi}} e^{\frac{-(x-\theta)^2}{2}}, \ -\infty < x < \infty$$

   Find the best critical region for testing $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ assume

$\theta_0 > \theta_1$.

7. Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables with distribution P_θ where θ is a real valued parameter. Suppose that $E_\theta X = g(\theta)$ and $Var_\theta(X) = \tau^2(\theta) < \infty$ where g is continuously differentiable function with derivative $g^{(1)}(\theta) > 0$ for all θ. Show that the estimator $\hat{\theta}$ obtained by solving the equation $g(\theta) = \bar{X}$ where $\bar{X} = \sum_{i=1}^{n} X_i/n$ is consistent. Also derive its asymptotic distribution.
8. Let $X_1, X_2, \cdots, X_n$ be a sample from a distribution function $F(\cdot)$ with density $f(\cdot)$. Since

$$f(y) = \lim_{h \to 0} \frac{F(y+h) - F(y-h)}{2h},$$

one suggest the estimator

$$\hat{f}(y) = \lim_{h_n \to 0} \frac{\hat{F}_n(y+h_n) - \hat{F}_n(y-h_n)}{2h_n},$$

where

$$\hat{F}_n(t) = \frac{\text{Number of } X_i \leq t}{n}.$$

Find the condition on h_n to guarantee that $\hat{f}(y)$ is a consistent estimate of $f(y)$.