臺灣大學數學系

八十七學年度第一學期碩博士班資格考試試題

統計與機率

probability

1. In $(\Omega, \mathcal{F}, P)$, let $A_n \in \mathcal{F}$. The set $\{A_n \quad i.o.\}$ is defined as $\{w; w \in A_n$ for an infinite number of $n\}$. Show the following statements hold.
   1. $\sum_1^\infty P(A_n) < \infty$ implies $P(A_n \;\; i.o.) = 0$.
   2. If A_n are independent events, then $\sum_1^\infty P(A_n) = \infty$ implies $P(A_n \;\; i.o) = 1$.
2. For any random variable X and real number $r > 0$,
   1. Show that $E|X|^r = r\int_0^\infty t^{r-1} P(|X| \geq t)\, dt$.
   2. If $E|X|^r < \infty$, then $P(|X| \geq t) = o(t^{-r})$ as $t \to \infty$. (You can assume that (a) holds to solve this problem.)
   3. Comment on the connection of statement (b) with $r = 2$ and the Chebyschev inequality.
3. Let the distribution functions $F, F_1, F_2, \cdots$ possess respective characteristic functions $\phi, \phi_1, \phi_2, \cdots$. Show that the following three statements are equivalent:
   1. F_n converges in distribution to F;
   2. $\lim_n \phi_n(t) = \phi(t)$, each real t;
   3. $\lim_n \int g dF_n = \int g dF$, each bounded continuous function g.
4. Consider a two-state Markov chain. The variables $X_1, X_2, \cdots$ each take on the values 0 and 1, with the joint distribution determined by $P(X_1 = 1) = p_1$ and $P(X_{i+1} = 1 \mid X_i = 0) = \pi_0$ and $P(X_{i+1} = 1 \mid X_i = 1) = \pi_1$ of which we assume $0 < \pi_0, \pi_1 < 1$. Set $\bar{X}_n = \sum_{i=1}^n (X_i/n)$. Show that $\bar{X}_n$ is a consistent estimate of $p = \pi_0/(\pi_0 + \pi_1)$.

Statistics

1. Find the asymptotic distribution of $\hat{p}_n^2$ where $\hat{p}_n$ is the proportion of success of a

binomial distribution with n trials and the probability of success p.

2. In life-testing experiments, it is quite often that the experiment is terminated whenever the first r failures have occurred among n tested units. This scheme is usually referred to as Type II censored sampling. Suppose the survival time of a particular unit follows an exponential distribution $Exp(\theta)$ (i.e., $F(x) = 1 - \exp(-\theta_x)$). Derive the maximum likehood estimate of θ under Type II censored sampling and discuss whether the resulting estimator is consistent when $r = 1$.