臺灣大學數學系

八十六學年度第二學期碩博士班資格考試試題

統計與機率

1. (15 points) Let Y be a number uniformly distributed over the unit interval $(0,1)$. Let $X_1, X_2, \cdots$ be the successive digits in the decimal expansion of Y, that is,

$$Y = \frac{X_1}{10} + \frac{X_2}{10^2} + \cdots + \frac{X_n}{10^n} + \cdots.$$

   1. (7 points) Prove that $X_1, X_2, \cdots$ are independent discrete valued random variables uniformly distributed over the integers 0 to 9.
   2. (8 points) Assume (a) holds. Show that for any integer $k\ (0 \le k \le 9)$, the set of numbers Y for which the relative frequency of k in the decimal expansion of Y is $1/10$ has probability 1. Does this contradict the fact that only three occur in the decimal expansion of $1/3$?

2. (10 points) Give a proof to show that if $E(X_n) \to 0$ and $Var(X_n) \to 0$, then $X_n \xrightarrow{P} 0$.

3. (10 points) Let X_1, X_2 be independent, identically distributed random variables with common uniform distribution on $(0,1)$. Let $X_{(2)} = \max(X_1, X_2)$. Find $P(X_1 \le x \mid X_{(2)} = y)$ and $E(X_1 \mid X_{(2)} = y)$.

4. (15 points) Let $X_1, X_2, \cdots$ be a sample from a $N(\mu, 1)$ population. Find the UMVU estimate of $P(X_1 \ge 0) = \Phi(\mu)$. Here Φ denotes the distribution function of a standard normal random variable.

5. (30 points) Let $X_1, \cdots, X_n$ be independent random variables, $X_i \sim Poisson(\lambda_i)$, $i = 1, \cdots, n$.
   1. (10 points) Deduce that the likelihood-ratio statistic for $H_0 : \lambda_1 = \cdots = \lambda_n$ versus $H_a : \lambda_i \ne \lambda_j$ for some i, j, is given by

$$2 \log \Lambda = 2 \sum_{i=1}^{n} X_i \log(\frac{X_i}{\bar{X}}),$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$.

2. (10 points) Explain why $-2\log\Lambda$ can be approximated by

$$\frac{1}{\bar{X}}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

3. (10 points) If H_0 is true and $\lambda_1, \cdots, \lambda_n \to \infty$, it is claimed that $2\log\Lambda$ has approximately the chi-square distribution with $n-1$ degrees of freedom. Please show that the above claim is correct. (You can assume (b) holds to proceed your proof.)

6. (20 points) Given the general linear model

$$E(Y \mid \mathcal{X}, \mathcal{Z}) = \mathcal{X}\beta + \mathcal{Z}\gamma; \qquad Var(Y) = \mathcal{I}\sigma^2$$

where Y is $n \times 1$, $\mathcal{X}$ is $n \times p, \beta$ is $p \times 1$, $\mathcal{Z}$ is $n \times p$, γ is $q \times 1$. Suppose that we believe that Y is not related to $\mathcal{Z}$. (In fact, Y is related to $\mathcal{Z}$ via the above model with $\gamma \neq 0$.) We fit the model

$$E(Y \mid \mathcal{X}) = \mathcal{X}\beta; \qquad Var(Y) = \mathcal{I}\sigma^2.$$

1. (10 points) What is the effect of the false model assumption on the estimation of β?
2. (10 points) Suppose that $p = 2, q = 1$, the ith element of $\mathcal{Z}$ is x_{ni}^2, and the ith row of $\mathcal{X}$ is $(1, x_{ni})$. Re-do (a) as $n \to \infty$ under the assumptions that the unobserved noises are normally distributed with mean 0 and variance σ^2, and that $x_{ni} = i/n$.