台灣大學數學系

九十二學年度第二學期博士班資格考試題

機率論

May 8, 2004

(1)

(20 points) **(1A)** Let (X_n) be a sequence of random variables. Prove that $X_n \to X$ in probability if and only if for every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ that converges to X almost surely (a.s.). **(1B)** Prove that if f is continuous and $X_n \to X$ in probability then $f(X_n) \to f(X)$ in probability.

(2)

(12 points) Let (X_n) be a sequence of i.i.d. random variables with $X_n > 0$. Let $S_n = X_1 + \cdots + X_n$ and $N(t) = \sup\{n : S_n \le t\}$. Prove that $\frac{N(t)}{t} \to \frac{1}{\mu}$ a.s. as $t \to \infty$, where $\mu = E[X_1] \in (0, \infty]$ and $1/\infty = 0$.

(3)

(36 points) **(3A)** Show that if random variables $X_n \to X$ in probability then X_n converges weakly to X (denoted as $X_n \Rightarrow X$). Conversely, show that if $X_n \Rightarrow c$, where c is a constant, then $X_n \to c$ in probability. **(3B)** Show that if $X_n \Rightarrow X$ and $0 \le Y_n \Rightarrow c$, where c is a constant, then $X_n Y_n \Rightarrow cX$. **(3C)** Let (X_n) be a sequence of i.i.d. random variables with $E[X_1] = 0$ and $E[X_1^2] \in (0, \infty)$. Prove that $\sum_{k=1}^{n} X_k \Big/ \left(\sum_{k=1}^{n} X_k^2 \right)^{1/2}$ converges weakly to a standard normal distribution.

(4)

(20 points) **(4A)** Let $\mathfrak{F}_n, n \ge 1$, be a filtration and (Z_n) be a sequence of integrable random variables adapted to $\mathfrak{F}_n$. Prove that Z_n can be written in a unique way as $Z_n = M_n + A_n$ such that $(M_n, \mathfrak{F}_n)$ is a martingale and A_n is a predictable sequence with $A_1 \equiv 0$. Moreover, (A_n) is increasing if and only if (Z_n) is a sub-martingale. **(4B)** Let (X_n) be a sequence of independent random variables with $E[X_n] = 0$ and $E[X_n^2] < \infty$ for all $n \ge 1$. Let

$\mathfrak{F}_n = \sigma(X_1, ..., X_n)$. Show that $Z_n = (X_1 + \cdots + X_n)^2$ is a sub-martingale and find M_n, A_n of the decomposition $Z_n = M_n + A_n$ described in **(4A)**.

(5)

(12 points) Given a probability space $(\Omega, \mathfrak{F}, P)$ and an integrable random variable X. Show that the family $\{E[X \mid \mathfrak{G}] : \mathfrak{G}$ is a $σ#sigma;$-field $\subset \mathfrak{F}\}$ is uniformly integrable.