台灣大學數學系

九十二學年度第二學期博士班資格考試題

偏微分方程

May 8, 2004

1.

(20%) Consider the following initial value problem

$$\begin{cases} u_{tt} - \sum_{i,j=1}^{3} a^{ij} u_{x_i x_j} = 0, & t > 0,\ x \in \mathbb{R}^3 \\ u(x,0) = 0, \quad u_t(x,0) = \phi(x), \end{cases}$$

where the matrix $A = (a^{ij})_{i,j=1}^{3}$ is a positive-definite symmetric matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \lambda_3$.

(a)

Write down the explicit formula of $u(x,t)$ (in integral form).

(b)

Assume supp $\phi \subset \{x : |x| \le \alpha\}$. For $t > 0$, estimate the support of $u(\cdot, t)$.

2.

(20%) Consider the wave equation

$$u_{tt} - \Delta u = 0 \quad (x,t) \in (0,\infty) \times \mathbb{R}^n.$$

Let $B(x_0, t_0)$ be the ball of radius $t_0 > 0$ centered at x_0 and $C := \{(x,t) : 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}$. Using the *energy estimate* without referring to the explicit formula of the solution to show that if $u(x,t) = u_t(x,t) \equiv 0$ in $B(x_0, t_0)$ then $u \equiv 0$ in C. Argue that the speed of propagation is finite. Can the strong maximum principle hold for the wave equation?

3.

(20%) Solve the Cauchy problem: $u_y = u_x^3$ with $u(x,0) = 2x^{3/2}$.

4.

(30%) (a) Let $U := \{u(x) \in C_0^\infty(\mathbb{R}^n) :$ supp $(u) \subset B(0, r_0)$ and $\lim_{r\to 0} e^{r^{-s}} u = 0$ for every $s > 0\}$, where $r = |x|$ and $r_0 < 1$. Derive the following

estimate: there exists a constant $C>0$ such that for all $u \in U$ and sufficiently large s

$$s^4 \int_{\mathbb{R}^n} r^{-2s-2} e^{2r^{-s}} |u|^2 dx \le c \int_{R^n} r^{s+2} e^{2r^{-s}} |\Delta u|^2 dx.$$

(Hint: set $v = e^{r^{-s}} u$ and use the integration by parts).
(b) Let $u \in H^2_{loc}(\mathbb{R}^n)$ satisfy the differential inequality

$$|\Delta u| \le M|u| \quad \text{for } x \in \Omega,$$

where $M>0$ and Ω is an open neighborhood of 0. Assume that

$$\lim_{r\to 0} e^{r^{-s}} u = 0 \quad \text{for every } s>0.$$

Show that u vanishes near 0. (Hint: use (a) on θu, where θ is a suitable cut-off function with $\theta(x)=1$ for $|x| \le r_0/2$ and $\theta(x)=0$ for $|x| \ge r_0$).

5.

(10%) Derive the explicit formula of the solution $u(x,t)$ to

$$\begin{cases} u_{tt} - u_{xx} = p(x,t) & x>0,\ t>0 \\ u(x,0) = f(x),\ u_t(x,0) = g(x) & x \ge 0 \\ u(0,t) = h(t) & t \ge 0, \end{cases}$$

where $f(0)=h(0)$ and $g(0)=h'(0)$. (Hint: divide the first quadrant by the line $x=t$).