臺灣大學數學系

九十學年度第二學期碩博士班資格考試試題

微分方程式

1.

(10 points) The well-known *Tricomi* equation takes the form

$$\mathbf{u}_{yy} - y u_{xx} = 0, \qquad \text{for} \qquad (x, y) \in \mathbb{R}^2. \tag{1}$$

(1)

If $y > 0$, find the characteristic equations and the associated characteristic curves for (1).

2.

(25 points) Consider the initial-boundary value problem of the damp wave equation

$$\begin{cases} u_{tt} + 2k u_t = c^2 u_{xx}, & \text{for} 0 < x < L, t > 0, \\ u(x,0) = f(x), u_t(x,0) = 0, & \text{for} 0 \le x \le L, \\ u(0,t) = 0, u(L,t) = 0 & \end{cases}$$

(2)

where k and c are positive real constants. If we define the total energy E as

$$E(t) = \frac{1}{2} \int_0^L \left[(u_t)^2 + (c u_x)^2 \right] \, dx.$$

(a) Show that the energy is decreasing, i.e., $dE/dt \le 0$.

(b) Use the fact from (a) to prove the uniqueness of the solution for (2).

3.

(25 points) Consider the Laplace equation in polar coordinates:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \tag{2}$$

(3)

(a) Solve (3) in a circular region: $0 \le r \le R$, $0 \le \theta \le 2\pi$ with the Neumann boundary condition $u_r = f(\theta)$ on the boundary of the circle.

(b) What condition should f be satisfied in order to have a existence and uniqueness of the solution for the problem ?

4

(25 points) Suppose that $u(X, t)$ is a smooth solution of the heat equation in space

$u_t = k\Delta u$ for $X \in \Omega,\ t \geq 0$, where Ω is a bounded region, and k is a positive constant.

(a) Suppose that we have Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Prove

$$\int_\Omega u(X,t)^2 dX \qquad \text{decays exponentially in time.}$$

(b) Suppose that we have Neumann boundary condition $u_N = 0$ on $\partial\Omega$, where N is the unit-outward normal to $\partial\Omega$. Prove

$$\int_\Omega u(X,t) dX \qquad \text{is constant in time.}$$

5.

(25 points) Let $u(x,t)$ be a solution of class C^2 of

$$u_t = a(x,t)u_{xx} + 2b(x,t)u_x + c(x,t)u$$

in the rectangle

$$\Omega = \{(x,t)|0 \leq x \leq L, 0 \leq t \leq T\}.$$

Let $\partial' \Omega$ denote the ``lower boundary'' of Ω consisting of the three segments

$$\begin{aligned} x = 0, &\quad 0 \leq t \leq T \\ 0 \leq x \leq L, &\quad\quad t = 0 \\ x = L, &\quad 0 \leq t \leq T. \end{aligned}$$

(a) Prove that in case $c < 0$ in Ω

$$|u(x,t)| \leq \sup_{\partial'\Omega}|u| \qquad \text{for} \quad (x,t) \in \Omega.$$

(b) Prove that more generally

$$|u(x,t)| \leq e^{CT}\max_{\partial'\Omega}|u|,$$

where

$$C = \max\left(0, \max_\Omega|c|\right).$$