# 臺灣大學數學系

# 八十九學年度第二學期碩博士班資格考試試題

# 微分方程式

Total score: $130$ points

**1.**

(20 points) Find the solution $u(x,t)$ in each case:

$$\begin{array}{ll}(a) & u_t - xuu_x = 0, \quad u(x,0) = x, \\ (b) & u_t + uu_x = 0, \quad u(x,0) = x^2.\end{array}$$

**2.**

(25 points) Consider the *telegraph equation* of the form

$$u_{xx} - c^{-2}u_{tt} - au_t - bu = 0, \qquad \textbf{(1)} \qquad (1)$$

where $a$, $b$, and $c$ are nonzero real constants. For a special class of problems, one may use the following simple approach to find the solution of the problems. Of course, since the equation is linear, alternatively, we may also use other standard method such as the Fourier transform method to solve the equation, but the solution procedure will be rather complicated in general.

(a)

Let $u(x,t) = w(t)v(x,t)$, show that $v$ satisfies the equation

$$v_{xx} - c^{-2}v_{tt} + kv = 0, \qquad \text{with} \quad k = b - \frac{a^2c^2}{4}, \qquad \textbf{(2)} \qquad (2)$$

where $u$ is a solution of **(1)** and $w$ is a solution of the equation

$$2c^{-2}\frac{dw}{dx} + aw = 0.$$

(b)

If $k = 0$, write down the general solution of **(1)** for a Cauchy problem with the initial conditions: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$.

(c)

If $k \neq 0$, and we look for solutions of **(2)** of the traveling wave form

$v = \phi(x - \gamma t)$, for some smooth function $\phi$. In this case, show that the solution

of **(2)** exists only when $\gamma \neq c$.

**3.**

(20 points) Show that if $u(X,t)$ of the form

$$u(X,t) = \begin{cases} \frac{1}{2}\, v(X,t)\, [t - \gamma(X)] & \text{for} \quad \gamma(X) \le t \\ 0 & \text{for} \quad \gamma(X) \ge t, \end{cases}$$

solves the wave equation in space,

$$u_{tt} - c^2 \Delta u = 0, \qquad X \in \mathbb{R}^3,$$

then the surface $S = \{t = \gamma(X)\}$ must be characteristic. That is, it satisfies the eikonal equation $|\nabla \gamma| = 1/c$. Here $v(X,t)$ is a $C^2$ function nonzero on the surface, and satisfies the transport equation

$$v_t + c^2 \nabla\gamma \cdot \nabla v = -\frac{1}{2} c^2 \left(\Delta \gamma\right) v.$$

In addition, show that $u(X,t)$ in this case is only a $C^1$ function.

**4.**

(25 points) Consider the Laplace equation in polar coordinates:

$$\Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0. \qquad (3)$$

**(3)**

(a)

Solve **(3)** in a circular region: $0 \le r \le R$, $0 \le \theta \le 2\pi$ with the Dirichlet boundary condition $u = f(\theta)$ on the boundary of the circle.

(b)

Show that $\lim_{r \to R} u(r,\theta) = f(\theta)$, where $u$ and $f$ are the solution and boundary condition in (a), respectively. Is $u(R,\theta) = f(\theta)$ true also ?

**5.**

(20 points) Assume that $f$ and $g$ are sufficiently smooth functions, show that the Robin problem:

$$\begin{array}{ll} \Delta u + a u = f(x,y), & \text{for} \quad (x,y) \in \Omega \\ \frac{\partial u}{\partial n} + b u = g(x,y), & \quad \text{on} \quad \partial\Omega, \end{array}$$

has a unique solution if $a < 0$ and $b > 0$.

**6.**

(20 points) Find a solution $u(x,t)$ of the one-dimensional heat equation $u_t - u_{xx} = 0$

in the quadrant $x > 0,\ t > 0$ satisfying the conditions: $u(x, 0) = 0$ and $u(0, t) = h(t)$.