臺灣大學數學系

八十八學年度第二學期碩博士班資格考試試題

微分方程式

Choose 4 out of the following 7 problems

1. Consider the one-dimensional wave equation $u_{tt}=c^2u_{xx}$ in the quadrant $x>0$, $t>0$, for which

$$\begin{aligned} u=f(x), \quad u_t=g(x) \qquad & \text{for} \quad t=0, \quad x>0 \\ u_t=\alpha u_x \qquad & \text{for} \quad x=0, \quad t>0, \end{aligned}$$

where f and g are of class C^2 for $x>0$ and vanish near $x=0$.

   1. Let α be a constant $\neq c$. Find the solution $u(x,t)$.
   2. Show that generally no solution exists when $\alpha=-c$.

2. Let $u(x,y)$ be a solution of a quasi-linear equation of the form

$$a(u_x,u_y)u_{xx}+2b(u_x,u_y)u_{xy}+c(u_x,u_y)u_{yy}=0.$$

Introduce new independent variables ξ, η and a new unknown function ϕ by

$$\xi=u_x(x,y), \qquad \eta=u_y(x,y), \qquad \phi=xu_x+yu_y-u.$$

   1. Prove that ϕ as a function of ξ and η satisfies $x=\phi_\xi$, $y=\phi_\eta$ and the linear differential equation

$$a(\xi,\eta)\phi_{\eta\eta}-2b(\xi,\eta)\phi_{\xi\eta}+c(\xi,\eta)\phi_{\xi\xi}=0.$$

   2. [(b)] As an application of the technique discussed in (a), find the general solution of the equation

$$u_x^2+u_yx=0.$$

3. Recall that the fundamental solution of Laplace's equation in space is

$$K(X,\xi)=-\frac{1}{4\pi|X-\xi|}.$$

The Green's function $G(X,\xi)$ for the Dirichlet problem in a bounded domain $\Omega\subseteq\mathbb{R}^3$ has the form

$$G(X,\xi)=K(X,\xi)+h(X,\xi),$$

where

$$\Delta h = 0 \quad \text{in} \quad \Omega, \qquad G(X,\xi) = 0 \quad \text{for} \quad X \in \partial\Omega.$$

Show that $G(X,\xi) = G(\xi, X)$. That is, G is a symmetric function of the two variables X and ξ.

4. Solve the Poisson equation in two space dimensions

$$u_{xx} + u_{yy} = \phi(x,y)$$

in the quarter plane with the Neumann boundary conditions

$$u_y(x,0) = f(x), \quad u_x(0,y) = g(y).$$

Give comments to f, g, and ϕ that should be satisfied in order for the solution of the problem to exist.

5. Show that if $u(X,t)$ of the form

$$u(X,t) = \begin{cases} \frac{1}{2}\, v(X,t)\,[t - \gamma(X)] & \text{for} \quad \gamma(X) \le t \\ 0 & \text{for} \quad \gamma(X) \ge t, \end{cases}$$

solves the wave equation

$$u_{tt} - c^2 \Delta u = 0$$

in space, $X \in \mathbb{R}^3$, then the surface $S = \{t = \gamma(X)\}$ must be characteristic. That is, it satisfies the eikonal equation

$$|\nabla\gamma| = 1/c.$$

Here $v(X,t)$ is a C^2 function nonzero on the surface, and satisfies the transport equation

$$v_t + c^2 \nabla\gamma \cdot \nabla v = -\frac{1}{2} c^2 \,(\Delta\gamma)\, v.$$

In addition, show that $u(X,t)$ in this case is only a C^1 function.

6. Let $u(x,t)$ be a positive solution of class C^2 of

$$u_t = \mu u_{xx} \quad \text{for} \quad t > 0.$$

   1. Show that $\theta = -2\mu u_x/u$ satisfies Burgers' equation

$$\theta_t + \theta\theta_x = \mu\theta_{xx} \quad \text{for} \quad t > 0.$$

   2. For $f \in C_0^2(\mathbb{R})$, find a solution of the Burgers' equation with initial values $\theta(x,0) = f(x)$.

7. Suppose that $u(X,t)$ is a smooth solution of the heat equation in space $u_t = \mu\Delta u$ for

$X \in \Omega$, $t \geq 0$, where Ω is a bounded region. Assume that Neumann boundary condition $u_N = 0$ on $\partial\Omega$, where N is the unit-outward normal to $\partial\Omega$. Show that

$$\frac{d}{dt}\left[\int_\Omega u(X,t)dX\right] = 0.$$

That is, $\int_\Omega u(X,t)dX$ is constant in time.