臺灣大學數學系

八十七學年度第一學期碩博士班資格考試試題

微分方程式

20 points each.

1. Given two continous functions f_1, f_2, solve the following hyperbolic linear system:

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \end{pmatrix}$$

$$\text{(i)} \begin{cases} u_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_x = \binom{1}{2} & t > 0,\ x \in \mathbb{R} \\ u(x,0) = \binom{f_1(x)}{f_2(x)}; \end{cases}$$

$$\text{(ii)} \begin{cases} u_t + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u_x = \binom{1}{2} & t > 0,\ x > 0 \\ u(x,0) = \binom{f_1(x)}{f_2(x)} \\ u_1(0,t) = u_2(0,t) + 1 & t > 0.\text{(This is a mixed problem.)} \end{cases}$$

2. (i) Show that the function $G(x,y)$ defined by

$$G(x,y) = \begin{cases} 1 & \text{for } x > a,\ y > b \\ 0 & \text{for all other } x, y \end{cases}$$

is a fundamental solution with the pole (a,b) of the operator $L = \partial^2 / \partial x \partial y$ in the $x - y$ plane. (i.e. $LG = \delta_a \delta_b$, δ is the Dirac function.)

(ii) Solve

$$\begin{cases} \partial^2 / \partial x \partial y = F(x,y) \\ u(x,x) = 0 \\ u_x(x,x) = u_y(x,x) = 0 \text{(Here initial datas are given on the line} \{(x,y) : x = y\}. \end{cases}$$

(iii) Using (ii), solve the following non-homogeneous equations:

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & u = u(x,t) \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) & \text{(Here } f, g \text{ are continous functions.)} \end{cases}$$

3. Let $\Omega = \{(x, y) : x > 0, y \in \mathbb{R}\}$. Solve the following Dirlichlet problem by constructing a Green's function:

$$\begin{cases} u_{xx} + u_{yy} = 0 \text{ on } \Omega \\ u(0, y) = f(y), f \text{ is bounded and continous.} \end{cases}$$

4. Let Ω be an open set in C. Assume f is a conformal map from Ω to Ω'. (i.e. f is analytic in Ω. $f'(z) \neq 0$ for all comples number z in Ω.) Let $f(x + iy) = u(x, y) + iv(x, y)$ and U is harmonic in Ω'. Let $V(x, y) = U(u(x, y), v(x, y))$. Prove that V is harmonic in Ω. i.e. $(\partial / \partial x)^2 V + (\partial / \partial y)^2 V = 0$ on Ω.

5. (i) Find solutions of $u(x, t)$ of the one-dimentional heat equation $u_t = u_{xx}$ of the form $u = (1 / \sqrt{t}) f(x / 2\sqrt{t})$.(f has to satisfy a linear second-order O.D.E..)

   (ii) Find solutions of $u(x, t)$ of the one-dimensional Burgers' equation $u_t + uu_x = u_{xx}$ of the form $u(x, t) = (1 / \sqrt{t}) f(x / \sqrt{t})$.