臺灣大學數學系

八十六學年度第一學期碩博士班資格考試試題

微分方程式

1. Solve the following PDEs.
    1. $yu_x + zu_y + u_z = zu^3, u(x,y,0) = e^{x-y}$.
    2. $u_y + u_x^2 + u = 0, u(x,0) = x$.
2. Let $\phi(x) = [\sin(2\pi x)]^4$. Assume u satisfies

$$\begin{aligned} &u_{tt} - u_{xx} = 0 \text{ for } -1 < x < 1, 0 < t, \\ &u(-1,t) = u(1,t) = 0 \text{ for } 0 < t, \\ &u(x,0) = \phi(x), u_t(x,0) = -\phi_x(x) \text{ for } x \in [0, \tfrac{1}{2}], \\ &u(x,0) = u_x(x,0) = 0 \text{ for } x \in [-1,0] \cup [\tfrac{1}{2}, 1]. \end{aligned}$$

   Find the functions $u(x,2)$ and $u(x,4)$.

3. (Liouville's theorem) Prove that a harmonic function defined and bounded in all of $\mathbb{R}^n$ is a constant.
4. Let $\Omega = \{(x,t) : -1 < x < 1, t > 0\}$ and let $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ on Ω satisfy

$$\begin{aligned} &u_t = u_x x \text{ on } \Omega \\ &u(-1,t) = u(1,t) = 0 \text{ for } t > 0, \\ &u(x,0) = f(x) \text{ for } -1 \le x \le 1. \end{aligned}$$

   Assume $f(-1) = 0$, $f(-x) = f(x)$ and $0 \le f(x) \le 1$.

    1. Show that $0 \le u(x,t) \le 1$ on Ω.
    2. Show that $u(x,t) = u(-x,t)$ on Ω.
    3. Prove that the energy $E(t) = \int_{-1}^{1} u^2(x,t)\,dx$ is decreasing in t for $t > 0$.

5. Let Ω denote an open bounded set in n-dimensional x-space described by an inequality $\phi(x) > 0$, so that $\phi(x) = 0$ on $\partial\Omega$. Let S_λ for $\lambda > 0$ denote the hyper-surface in xt -space given by $t = \lambda\phi(x)$ for $x \in \Omega$. On S_λ define

$$E(\lambda) = \int_{S_\lambda} Q_\lambda \, dx,$$

   where

$$Q_\lambda = \frac{1}{2}(u_t^2 + c^2 \sum_i u_{x_i}^2) + \lambda c^2 u_t \sum_i u_{x_i} \phi_{x_i}.$$

1. Prove $E(\lambda) =$ constant when $u_{tt} - c^2 u_{xx} = 0$.
2. Show that Q_λ as a quadratic form in $u_t, u_{x_1}, \ldots, u_{x_n}$ is positive definite, when S_λ is spacelike.
3. Show that the initial data on S_0 of a solution of $u_{tt} - c^2 u_{xx} = 0$ uniguely determine u on all S_λ with sufficiently small λ.

6. Let $f(\theta)$ be a C^4 function of period 2π with Fourier series

$$f(\theta) = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)].$$

1. Prove that

$$u = \sum_{n=0}^{\infty} [a_n \cos(n\theta) + b_n \sin(n\theta)] r^n$$

represents in polar coordinates r, θ the solution of the Laplace equation $\Delta u = 0$ in the disk $x^2 + y^2 < 1$ with boundary values f.

2. Derive Poisson's integral formula

$$u(r, \theta) = \frac{1 - r^2}{2\pi} \int_0^{2\pi} \frac{f(\alpha)}{1 + r^2 - 2r\cos(\theta - \alpha)} d\alpha$$

from (a) by substituting for the a_n, b_n their Fourier expressions in terms of f and interchanging summation and integration.