台灣大學數學系

九十二學年度第二學期博士班資格考試題

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1.

(20 %) (a) Suppose that $(A_1, A_2, \ldots, A_n)$ is a family of sets such that $|A(J)| \geq |J|$ for all $J \subseteq \{1, 2, \ldots, n\}$, where $A(J) = \cup_{j \in J} A_j$. Prove that if $|A_i| \geq r$ for $1 \leq i \leq n$, then the number of SDRs for this family is at least $r!$ if $r \leq n$, and is at least $r(r-1)\ldots(r-n+1)$ if $r > n$.

(b)

Given a $k \times n$ Latin rectangle with $k < n$, **prove that** there are at least $(n-k)!$ ways to add a row to form a $(k+1) \times n$ rectangle.

2.

(20 %) (a) Suppose (A_{ij}) is an $n \times n$ real matrix whose entries satisfy $|a_{ij}| \leq 1$ for all i, j. Prove that $|\det(A)| \leq n^{n/2}$; and the equality holds if and only if $a_{ij} = \pm 1$ for all i, j and $AA^{\mathsf{T}} = nI$. (Such a matrix is called an *Hadamard matrix*.)

(b)

Prove that if a Hadamard matrix of order n exists, then either $n = 1$ or $n = 2$, or $n \equiv 0$ (mod 4).

3.

(20 %) Let A and B be two $m \times n$ matrices with entries in $\{0, 1\}$. An *exchange* operation substitutes a submatrix of the form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for a submatrix of the form $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or vice versa. Prove that if A and B have the same list of row sums and have the same list of column sums, then A can be transformed into B by a sequence of exchange operations. Interpret this conclusion in the context of bipartite graphs.

4.

(20 %) Let G be a connected graph of n vertices. Define a new graph G' having one vertex for each spanning tree of G, with vertices adjacent in G' if and only if the corresponding

trees have exactly $n-2$ common edges. Prove that G' is connected. Determine the diameter of G'.

5.

(20 %) The k th *power* of a simple graph G is the simple graph G^k with vertex set $V(G)$ and edge set $\{uv : 1 \leq d_G(u,v) \leq k\}$. (a) Suppose that $G-x$ has at least three nontrivial components in each of which x has exactly one neighbor. Prove that G^2 is not Hamiltonian. (b) Prove that the cube of each connected graph (with at least three vertices) is Hamiltonian. (HINT: Reduce this to the special case of trees, and prove a stronger result that if xy is an edge of the tree T, then T^3 has a Hamiltonian cycle using the edge xy.)