台灣大學數學系
九十一學年度第二學期博士班資格考試題
離散數學

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1. (17 %) Suppose A is an n-set and $f_m = |\mathcal{F}_m|$ for $0 \leq m \leq 2$, where $\mathcal{F}_m = \{B : B \subseteq A$ and $|B| \equiv m$ (mod 3)$\}$. Determine f_0, f_1 and f_2.

2. (17 %) Recall that Ramsey's theorem is as follows. Suppose $a_1, a_2, \ldots, a_r, t$ are positive integers such that each $a_i \geq t$. There exists a positive integer n with the following property: If all the t-subsets of an n-set are colored with r colors, then there is an i and an a_i-subset all of whose t-subsets are colored by the ith color. We denote by $R(a_1, a_2, \ldots, a_r; t)$ the smallest n for which Ramsey's theorem holds. (2-1) Prove that for any five points in the plane, no three colinear, some four of the points form a convex quadrilateral. (2-2) Prove that for a set of n points in the plane, if every four points form a convex quadrilateral, then all n points form a convex polygon. (2-3) Prove that for any $R(n, 5; 4)$ points in the plane, no three colinear, some set of n of the points form a convex polygon.

3. (16 %) Let $d_1, d_2, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there exists a tree with vertex degrees $d_1, d_2, \ldots, d_n$ if and only if $\sum_{i=1}^{n} d_i = 2n - 2$.

4. (17 %) A *doubly stochastic matrix* is a nonnegative real matrix in which every row and every column sums to 1. Prove that a doubly stochastic matrix Q can be expressed $Q = c_1P_1 + c_2P_2 + \ldots + c_mP_m$, where $c_1, c_2, \ldots, c_m$ are nonnegative real numbers summing to 1 and $P_1, P_2, \ldots, P_m$ are permutation matrices. For example,

$$\begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 1/6 & 5/6 \\ 1/2 & 1/2 & 0 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} + \frac{1}{3}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

5. (17 %) Let G be a graph having no induced subgraph isomorphic to P_4. Prove that its chromatic number $\chi(G)$ is equal to its clique number $\omega(G)$. **6.** (16 %) The kth power of the n-path is the graph P_n^k whose vertex set is $V(P_n^k) = \{1, 2, \dots, n\}$ and edge set is $E(P_n^k) = \{ij : 1 \leq i < j \leq n$ and $|i-j| \leq k\}$. Determine all pairs (n, k) for which P_n^k is planar. Give reasons.