台灣大學數學系
九十一學年度第二學期博士班資格考試題
實分析

May 10, 2003

There are 12 problems,do any 10 of them.

Let $(X, \mathbf{B}, \mu)$ be a σ-finite measure space (i.e. there exist countably many $X_j \in \mathbf{B}$ such that $\mu X_j < \infty$ and $X = \cup_j X_j$).Let $f \in L^1\ (\mu)$

(1)Prove for any $\epsilon > 0$, there is a measurable function g vanishing outside a set of finite measure such that $\int |f-g| d\mu < \epsilon$.

(2)Prove for any $\epsilon > 0, \exists \delta > 0$ such that $\int_A |f| d\mu < \epsilon$ for any A $\in \mathbf{B}$ with $\mu A < \delta$.

(3)Define a signed measure ν by ν E= $\int_E f d\mu$. Let $\{E_n\}$ be a decreasing sequence of measurable sets. Prove $\nu(\cap_1^\infty E_j) = lim_{j\to\infty} \nu E_j$.

(4)If $X = [0,1]$ and μ is the Lebesgue measure , prove $F(x)$ defined by $F(x) = \int_0^x f(t)dt$ is absolutly continuous.

Let f be a real-valued measurable function on [a,b] and $E = \{x : |f'(x)| < \alpha\}$.Let m^* and m be respectively the Lebesgue outer measure and Lebesgue measure on [a,b].

(5) If f is absolutely continuous, prove $m^*(f(E)) \le \alpha m^*(E)$.

(6) If f is of bounded variation , is it still true that $m^*(f(E)) \le \alpha m^*(E)$?Prove or disprove your answer .

Let X be a compact subset of $\mathbf{R}^n$. Let $f_n \in L^p(X)(p > 1)$ be a sequence converging a.e. to a measurable function f. Suppose $\|f_n - f\|_p < M$ for some M .

(7) Prove $f \in L^p(X)$ and f_n converges to f weakly .

(8) Prove f_n converges to f in $L^r(X)$ for any $1 \le r \le p$.

Let μ be a measure on the compact subset K of $\mathbf{C}$. Define $f(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z}$.

(9) Prove f is analytic outside K .

(10) Find an estimate for $|f'(z)|$ for z not in K .

Let ϕ be a bounded measurable function defined on $\{(x,y) : |x|^2 + |y|^2 \leq 1\} \subset \mathbf{R}^2$.

Define for $(x, y) \in \mathbf{R}^2$

$$g(x,y) = \int_{|s|^2+|t|^2\leq 1} \phi(s,t) \log((s-x)^2 + (t-y)^2) dm,$$

where m is the Lebesgue measure on $\mathbf{R}^2$.
(11) Prove g is locally integrable hence finite a.e..

(12) Prove the partial derivatives of g exist .