臺灣大學數學系

九十學年度第一學期碩博士班資格考試試題

分析

1.

[a] Show that the function h defined by $h(x)=c^{-\frac{1}{x^2}}$, $x>0$; $h(x)=0$, $x\leq 0$ is in C^∞.

[b] Show that the function $g(x)=h(x-a)h(b-x)$, $a<b$ is C^∞ and with support in $[a,b]$.

[c] Construct a function in $C_0^\infty(R^n)$ whose support is a ball.

2.

Let $\phi(x)$ be a bounded positive measurable function such that $\phi(x)=0$ outside $[-1,1]$ and $\int\phi=1$. For $\varepsilon>0$ let $\phi_\varepsilon(x)=\varepsilon^{-1}\phi(\frac{x}{\varepsilon})$. Show that $\lim\limits_{\varepsilon\to 0}(f*\phi_\varepsilon)(x)=f(x)$ in the Lebesgue set of f. ($*$ denote convolution)

3.

[a] Show that in $L^2[0,1]$, the parallelogram law holds, ie.

$$||f+g||^2+||f-g||^2=2||f||^2+2||g||^2.$$

[b] Is it true for L^P, $p\neq 2$?

4.

Prove that together with $f_n(x)\geq 0$ and $If_n\to 0$, imply $f_n\to 0$ in measure, but (in general) not $f_n\to 0$ almost everywhere. Also, the condition $f_n(x)\geq 0$ cannot be dropped.

5.

[a] $1<p<q<\infty$, show that $L^P[0,1]\supset L^q[0,1]$ and that

$$||f||_p\leq ||f||_q \text{ for } f\in L^q.$$

[b] $\lim\limits_{p\to\infty}||f||_p=||f||_\infty$ for $f\in L^\infty[0,1]$.

6.

[a] Let f be analytic in a region D in upper half plane, the boundary ∂D intersect the real line in a interval $[a,b]$ (Fig. 1). f is continuous on $D\cup[a,b]$ and takes real values

on $[a, b]$. Show that (the Schwarz reflection principle) f can be continued analytically into D^* (reflection of D).

[b] What can you conclude about the case in Fig. 2, where $f(z) = z^{\frac{1}{2}}$ is defined on D?