臺灣大學數學系

八十八學年度第二學期碩博士班資格考試試題

分析(加考)

1. Let **Z** be a subset of $\mathbb{R}^1$ with measure zero. Show that the set $\{x^2 : x \in \mathbf{Z}\}$ also has measure zero.

2. If $\lambda_1 < \lambda_2 < \cdots < \lambda_m$ is a finite sequence and $-\infty < s < +\infty$, write $\sum_k a_k e^{-s\lambda_k}$ as a Riemann-Stieltjes integral. [Take $f(x) = e^{-sx}$, ϕ to be an appropriate step function, and $[a, b]$ to contain all the λ_k inits interior.]

3. Let $f(x, y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x, y)$ is an integrable function of y, and $(\partial f(x, y)/\partial x)$ is a bounded function of $f(x, y)$. Show that
$$\frac{d}{dx}\int_0^1 f(x, y)\, dy = \int_0^1 \frac{\partial}{\partial x} f(x, y)\, dy.$$

4. (a)
   Let $\{f_k\}$ be a sequence of measurable functions on E. Show that $\sum f_k$ converges absolutely a.e. in E if $\sum \int_E |f_k| < +\infty$. [Use theorem (5.16) and (5.22).]

   (b)
   If $\{r_k\}$ denotes the rational numbers in $[0, 1]$ and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k |x - r_k|^{-1/2}$ converges absolutely a.e. in $[0, 1]$.

5. Let E be a measurable subset of $\mathbb{R}^2$ such that for almost every $x \in \mathbb{R}^1$, $\{y : (x, y) \in E\}$ has $\mathbb{R}^1$-measure zero. Show that E has measure zero, and that for almost every $y \in \mathbb{R}^1$, $\{x : (x, y) \in E\}$ has measure zero.

6. Show that if $\alpha > 0$, x^α is absolutely continuous on every bounded subinterval of $(0, \infty)$.

7. Let $f, \{f_k\} \in L^p$. Show that if $\|f - f_k\|_p \to 0$, then $\|f_k\|_p \to \|f\|_p$. Conversely, if $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p, 1 \le p \le \infty$, show that $\|f - f_k\|_p \to 0$.

8. Prove the following generalization of Hölder's inequality. If $\sum_{i=1}^{k} 1/p_i = 1/r,\ p_i, r \ge 1$, then

$$\|f_1 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

9. For $f \in L(\mathbb{R}^1)$, define the *Fourier transform* $\hat{f}$ of f by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,dt \qquad (x \in \mathbb{R}^1).$$

(For a complex-valued function $F = F_0 + iF_1$ whose real and imaginary parts F_0 and F_1 are integrable, we define $\int F = \int F_0 + i \int F_1$.) Show that if f and g belong to $L(\mathbb{R}^1)$, then

$$\widehat{(f * g)}(x) = \hat{f}(x)\hat{g}(x).$$

10. If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $k \to \infty$, show that $f_k \xrightarrow{m} f$ on E (and thus that there is a subsequence $f_{k_j} \to f$ a.e. in E).