臺灣大學數學系

八十八學年度第二學期碩博士班資格考試試題

分析

Part A: There are 7 problems in this section, do any 4 problems from them.

1. $x \in (0,1)$, Call x a lucky number if there are infinite many 8's in the decimal expansion of x. Call x a really lucky number if there are infinitely many 8's, infinitely many 88's, infinitely many 888's, infinitely many 8888's … in the decimal expansion. Show that the set of really lucky number is measurable and find its measure.
2. $f \in L^1(a,b)$, given $\varepsilon > 0$, there exists $\delta > 0$ such that $\int_E | f(x) | dx < \varepsilon$ whenever $m(E) < \delta$ for all measurable subset $E \subset (a,b)$. Where $m(E)$ is the Lebesgue measure of E.
   True or false? Justify your answer.
3. Let $f(x) = \sum_{n=0}^{\infty} \frac{\cos nx}{2^n}$

   (a) Does $f \in C^0(R)$?

   (b) Does $f \in C^1(R)$?

   (c) Does $f \in C^\infty(R)$?

   (d) Is f analytic on R ?
4. Let $f \in L^1(0,\pi)$, find the following limits:

   (a) $\lim_{n\to\infty} \int_0^\pi f(x) \sin nx dx$

   (b) $\lim_{n\to\infty} \int_0^\pi f(x) \,|\sin nx\,|\, dx$
5. (a) Is the closed unit ball in l^2 compact ?
   (b) Let $E = \{(x_1, x_2, x_3, \cdots) \mid | x_i | \le \frac{1}{n}\}$. Is E compact in l^2 ?
6. Does there exist a bounded linear operator $T : L^2(0,1) \to L^2(0,1)$. Whose range is the set of all polynomials ?
7. Show that $L^P(0,1)$ is complete for $1 \le p < \infty$.

Part B: There are 4 problems in this section, do any 2 problems from them.

1. Evaluate the following integrals:

(a) $\int_{|z|=r} \frac{|dz|}{|z-a|^2}$ (b) $\int_{|z|=r} \frac{d\theta}{z-\alpha}$, $|\alpha| \neq r$ (c) $\int_{|z|\leq 1} \frac{dxdy}{z-\alpha}$

2. If $f(z)$ is analytic in $|z| \leq 1$ and satisfies $|f(z)| = 1$ on $|z| = 1$. Show that $f(z)$ is a rational function.
3. Let Ω be the unit disk minus the real interval $(0,1)$. Find a harmonic function u in Ω so that $u = 1$ on $|z| = 1$ and $u = 0$ on $(0,1)$.
4. Let C_1, C_2, be two circles in the plane and suppose that C_1 lies inside C_2. Let Γ_1 be a circle in the annular region which is tangent to C_1 and C_2 somewhere and let Γ_2 be the circle tangent to C_1, C_2 and Γ_1; Γ_3 be circle tangent to C_1, C_2 and Γ_2; $\cdots$.

   Suppose that the nth circle Γ_n tangent to Γ_1 (as shown in the figure).

   Show that no matter how the first circle Γ_1 take place in the annular region, the last circle Γ_n is always tangent to Γ_1 and n is independent of Γ_1.