臺灣大學數學系

八十七學年度第一學期碩博士班資格考試試題

分析(加考)

A.

Choose 4 from the following 6 problems

1.

(a)

Let $\{f_n\}$ be a sequence of measurable functions on $[0,1]$. Assume $\lim_{n\to\infty} f_n(x) = f(x)$ for all $x \in [0,1]$. Show that f is measurable.

(b)

Let f and g be two measurable functions. Show that fg is measurable.

2.

Assume $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. (a) Let $z \in \mathbb{R}^n$ and $f_z(x) = f(x+z)$. Show that

$$||f_z - f||_{L^p} \to 0 \text{ as } z \to 0.$$

(b) Let

$$g(y) = \int_{\mathbb{R}^n} f(x+y) e^{-|x|^2 - |y|^2}\, dx.$$

Is g continuous? Is g differentiable?

3.

Let $f, f_n \in L^p([0,1])$ with $1 \le p < \infty$. Show that if $f_n \to f$ a.e. and $||f_n||_{L^p} \to ||f||_{L^p}$ as $n \to \infty$, then $||f_n - f||_{L^p} \to 0$ as $n \to \infty$.

4.

Let f be of bounded variation on $[a,b]$.

(a) Show that $\frac{df}{dx}$ exists a.e. on $[a,b]$.

(b) Let $V[a,b]$ be the variation of f. Show that if

$$V[a,b] = \int_a^b |\frac{df}{dx}|\, dx,$$

then f is absolutely continuous on $[a,b]$.

5.

Find the value of p for which the function $f(x,y) = x^{-\frac{1}{3}}(x+y)^{-\frac{1}{5}}$ is in

$L^p([0,1]^2)$.

6.

Let $f \in C^1([0,1])$ and $Z = \{x \in [0,1] : \frac{df}{dx}(x) = 0\}$. Show that the set $f(Z) = \{f(x) : x \in Z\}$ has measure zero.

B.

Choose 1 from the following 2 problems

1.

Show that the image of an open set under a nonconstant analytic function is an open set.

2.

Evaluate the intrgral

$$\int_0^\infty \frac{dx}{1+x^n},$$

where $n \geq 2$ is a positive integer.