臺灣大學數學系

八十七學年度第一學期碩博士班資格考試試題

分析

(前六題任選四題, 後二題任選一題)

一.

a.

Let $\{f_n\}$ be a sequence of real-ralued Lebesgue integrable functions on an interval $[a,b]$. State and prove sufficient conditions for $\int_a^b \lim f_n(x)dx$ and $\lim \int_a^b f_n(x)dx$ to exist and be equal.

b.

Let $\{f_n\}$ be a sequence of real-ralued functions and continuously differentiable on $[a,b]$. State and prove sufficient conditions for $\lim f_n'(x)$ and $(\lim f_n)'(x)$ to exist and be equal for $x \in [a,b]$.

二.

If $f(x)$, $x \in R$, is continuous at almost every point of an interval $[a,b]$, show that f is measurable on $[a,b]$.

三.

a.

If f is measurable on $E \subset \mathbb{R}^n$, show that

$$|\{x \in E : f(x) > \alpha\}| \leq \frac{1}{\alpha^p} \int_{\{f>\alpha\}} f^p(x)\, dx\,,\ \alpha > 0,\ 0 < p < \infty\,.$$

b.

Let f and $\{f_n\}$ be measurable functions on E. If $p > 0$, and $\int_E |f - f_n|^p\, dx \to 0$ as $n \to \infty$, show that $f_n \xrightarrow{m} f$ on E (converge in measure on E).

四.

Let φ be a convex function on $(-\infty, \infty)$ and f an integrable function on $[0,1]$. Show that

$$\int_0^1 \varphi(f(t))\, dt \geq \varphi(\int_0^1 f(t)\, dt)\,.$$

五.

Let f be a Lebesgue integrable function on $[0,1]$ and assume that

$\int_0^1 f(x)\varphi(x)\ dx = 0$ for all continuous functions $\varphi : [0,1] \to R$. Show that $f = 0$ almost everywhere.

六.

Let φ be a positive continuous function on R satisfying $\varphi(x) = 0$ for $|x| > 1$ and $\int_{-\infty}^{\infty} \varphi(x)\ dx = 1$. Defint $\varphi_n(x) = n\varphi(nx)$, $n = 1, 2, \cdots$. Show that if f is a continuous function on R, then $\int_{-\infty}^{\infty} \varphi_n(x-y) f(y)\ dy$ converges to $f(x)$ as $n \to \infty$ for all $x \in R$.

七.

Let f be a function which is analytic in an open set containing the closed disc $D = \{z : |z| \leq 1\}$. Show that the points where $Re(f)$ (the real part of f) takes its maximum in D are on the boundary of D, and show the same for minimum points.

八.

Find the Laurent expansion for the function $f(z) = \frac{(z+2)e^z}{z^3}$ in powers of z. In what region of the complex plane does this series converge. Calculate the integral $\oint_{\{|z|=1\}} f(z)\ dz$, where the circle $|z| = 1$ is travelled in the counterclockwise direction.