Algebra Exam
Jun, 2005

Do all the 6 problems.

We use the follow notations:
$\mathbb{R}$: the field of real numbers.
$\mathbb{Z}$: the ring of integers.
$\mathbb{F}_q$: the finite field of q elements (where $q = p^n$ is a prime power).
C_n: cyclic group of order n.

(1) Determine how many irreducible polynomials are there of degree n over $\mathbb{F}_p$. And verify your answer.
(2) Determine the conjugacy classes of 8×8 matrices with minimal polynomial $(x^2+4)(x-1)^2$
  (a) over $\mathbb{R}$;
  (b) over $\mathbb{F}_5$.
(3) Describe the following as explicit as possible:
  (a) $Hom_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$
  (b) $\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$
(4) Give examples of polynomial $f(x) \in k[x]$ (and its ground field k) so that the Galois group of the polynomial $f(x)$ is
  (a) C_5;
  (b) S_5.
  And verify your examples.
(5) Let k be an algebraically closed field. Let $f(x,y)$ be an irreducible polynomial in $k[x,y]$. Describe prime ideals of the ring $k[x,y]/(f(x,y))$, where $(f(x,y))$ denotes the principal ideal generated by $f(x,y)$.
(6) Suppose that we have the following commutative diagram of abelian groups

$$\begin{array}{ccccc} A_1 & \xrightarrow{\phi_1} & A_2 & \xrightarrow{\phi_2} & A_3 \\ {\scriptstyle f_1}\downarrow & & {\scriptstyle f_2}\downarrow & & {\scriptstyle f_3}\downarrow \\ B_1 & \xrightarrow{\psi_1} & B_2 & \xrightarrow{\psi_2} & B_3. \end{array}$$

That is, all the above maps are group homomorphisms and $f_3\phi_2 = \psi_2 f_2, f_2\phi_1 = \psi_1 f_1$. Suppose furthermore that $im(\phi_1) \subset ker(\phi_2)$ and $im(\psi_1) \subset ker(\psi_2)$. Show that there is a natural map from $ker(\phi_2)/im(\phi_1)$ to $ker(\psi_2)/im(\psi_1)$