台灣大學數學系

九十二學年度第二學期博士班資格考試題

代數

May 8, 2004

Do 6 out of the following 8 problems. Do at least one problem in each section. If you do more than 6 problems, indicate which you want graded.
We use the follow notations:
$\mathbb{R}$: the field of real numbers.
$\mathbb{Q}$: the field of rational numbers.

$\mathbb{Z}$: the ring of integers.
$\mathbb{F}_q$: the finite field of q elements (where $q = p^n$ is a prime power).

C_n: cyclic group of order n.

1. Groups

(1)

Classify groups of order 20 up to isomorphism.

(2)

Is there any isomorphism between any two of the following groups? If there is an isomorphism, find an explicit one. If not, explain why.

(a)

S_4.

(b)

$S_3 \times C_4$.

(c)

$Q_8 \times C_3$.

(d)

$$SL(2, \mathbb{F}_3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} | a, b, c, d \in \mathbb{F}_3, ad - bc = 1 \right\}$$

2. Rings

(1)

Let K be a field and x, y, z, w are indeterminates. Show that $K[x, y, z, w]/(xy - zw)$ is **not** a UFD.

(2)

Let K be a finite extension over $\mathbb{Q}$. Let R be the integral closure of $\mathbb{Z}$ in K. Show that R is a free module over $\mathbb{Z}$ of rank $= [K : \mathbb{Q}]$.

3. Fields

(1)

Determine the Galois group of $x^5 - 2$

(a)

over $\mathbb{Q}$

(b)

over $\mathbb{F}_5$

(2)

Determine how many irreducible polynomials are there of degree 6 over $\mathbb{F}_p$. And verify your answer.

4. Linear Algebra

(1)

Determine the similarity classes (=conjugacy classes) of 6×6 matrices with minimal polynomial $(x^2+1)(x-1)^2$

(a)

over $\mathbb{R}$

(b)

over $\mathbb{F}_5$

(2)

Let V be an **even** dimensional vector space over $\mathbb{R}$ and $T : V \to V$ a linear transformation. Suppose that $T^3 = I$. Show that there is a linear transformation $S : V \to V$ such that $S^2 = -I, ST = TS$.