台灣大學數學系
九十一學年度第二學期博士班資格考試題
代數

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1.

By a rotation of the 3-dimensional real space R^3, we mean a rotation about a line through the origin. Let A be a linear transformation of R^3. Suppose that A preserves the Euclidean distance in the sense that $||Ax|| = ||x||$ for all $x \in bR^3$. (1) If the determinant of A is ≥ 0, show that A must be a rotation of bR^3. (2) Show that all rotaions of R^3 forms a group.

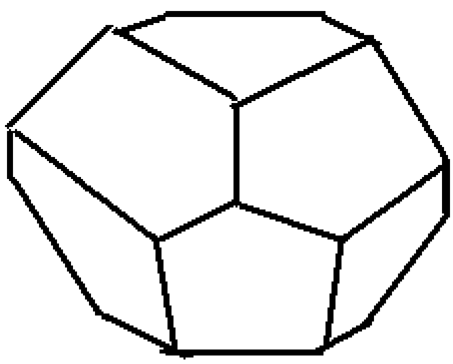
2.

Let G be the group of all rotations of R^3 which send a regular dodecahedron D, centered at the origin, to itself. Let G act on vertice of D. Find the number of orbits, the order of the stabilizer of a particular vertex and the order of G.

說明：Dodecahedron 是正十二面體，每一面都是正五邊形。Vertex (其複數是vertices)

表頂點，由 三個面相交而成，如下圖。一個頂點之stabilizer，是把此頂點固定

之所有 $g \in G$ 所成之集合。



3.

A subgroup H of a group G is said to be of finite index if $[G : H]$ is finite. (Note that H may *not* be normal.) (i) If H is a subgroup of finite index, show that there exists an integer N such that $g^N \in H$ for all $g \in G$. (ii) Let G be the group of invertible 2×2 matrices over the complex numbers. Find all subgroups of G which is of finite index.

4. Let R be the ring of all real-valued continuous functions defined on the unit disc $x^2+y^2\leq 1$ with the pointwise addition and multiplication. Find all maximal ideals of R.

5.

Let $\leq$ be a partial order defined on the set P. A subset C of P is called a *chain* if for any $a,b\in P$, either $a\leq b$ or $b\leq a$. A subset A of P is called an *antichain* if for any $a,b\in P$, neither $a\leq b$ nor $b\leq a$. If all chains and all antichains of P are finite, prove that P must be finite.

Note: A partial order $\leq$ on P is a binary relation $\leq$ satisfying the following: (i) Reflexivity: $a\leq a$ for $a\in P$. (ii) Transitivity: if $a\leq b$ and $b\leq c$, then $a\leq c$. (iii) Antisymmetry: if $a\leq b$ and $b\leq a$, then $a=b$.

6.

(i) Let R be a commutative ring with 1. Let J be the intersection of all maximal ideals of R. Prove that $a\in J$ if and only if $1-ar$ is invertible for all $r\in R$. (ii) Find the intersection of maximal ideals of the subring R of rationals defined by

$$R=\{\frac{m}{2n+1}\text{ : m , n are integers}\}.$$

7. Determine the Galois group of $x^4+3x^3-3x-2=0$ over the field of rationals.