臺灣大學數學系

九十一學年度第一學期碩博士班資格考試題

代數 (Algebra)

Sept 14, 2002

- (1)(15%) Let A be a real tridiagonal matrix (that is, an $n \times n$ matrix all of whose entries are zero except possibly those of the form $a_{i-1,i}, a_{ii}, a_{i,i-1}$. Show that if $a_{i,i-1}$ and $a_{i-1,i}$ are both positive, both negative, or both zero for $i = 2, \cdots, n$, then A has real eigenvalues. (*Hint*: A diagonal matrix D can be found such that DAD^{-1} is real symmetric.)

- (2)(15%) Let V be a finite dimensional vector space and $T : V \to V$ be a linear transformation. Let W be an invariant subspace (that is, $TW \subset W$). Let $m(t), m_1(t)$ and $m_2(t)$ be the minimal polynomial of T as linear transformation of $V, W, V/W$, respectively. Show that $m_i(t)|m(t), i = 1, 2$, and $m(t)|m_1(t)m_2(t)$.

- (3)(20%) Let N, H, K be normal subgroups of a group G. (a) Suppose $NK = HK$ and $N \cap K = H \cap K$. Is $N = H$ necessarily true? (b) Does G necessarily contain a subgroup isomorphic to G/N ? (c) Suppose $G/N \cong G/H$. Is $N \cong H$ necessarily true? (d) Suppose $G/N \cong G/H$ and $N \subseteq H$. Is $N = H$ necessarily true?

- (4)(15%) Let p be a prime. Show that for any $a \in Z/pZ$, there exist $b, c \in Z/pZ$ such that $a = b^2 + c^2$.

- (5)(15%) Describe all subrings (containing 1) of Q.

- (6)(20%) Let $K_0 = Q$. If $i \geq 0, K_{i+1}$ is the smallest subfield of C containing the set $\{a \in C | a^n \in K_i for some n > 0\}$. Let $K = \bigcup_{i=0}^{\infty} K_i$. (a) Prove that K is a field. (b) Let $f(x) \in K[x]$ be irreducible. Prove that deg $f(x) \neq 2, 3, 4$.