臺灣大學數學系

九十學年度第二學期碩博士班資格考試試題

代數

1.

(15 points) Let $\mathbb{N}$ denote the set of positive integers. Let F_i be a fields of characteristic $p_i > 0$, for all $i \in \mathbb{N}$. Suppose that $p_i \neq p_j$, for all $i, j \in \mathbb{N}$. Let $F = \prod_{i\in\mathbb{N}} F_i$ and $K = \oplus_{i\in\mathbb{N}} F_i$.

a. Show that K is an ideal of F. Is K a prime ideal of F?
b. Show that there exists a maximal ideal M of F such that $K \subseteq M$ and F/M is a field.
What is the characteristic of the field?

2.

(15 points) Show that there are no simple groups of order 1000. Find a non-abelian group of order 1000.

3.

(15 points) Let A be a complex $n \times n$ matrix. Show that there exist complex polynomials $p(t)$ and $q(t)$ with $p(0) = q(0) = 0$ with the properties such that $p(A)$ is diagonalizable, $q(A)$ is nilpotent and $A = p(A) + q(A)$.

4.

(15 points) Let R be a ring with identity and let $M_n(R)$ denote the ring of all $n \times n$ matrices with entries in R.
a. Find the center of $M_n(R)$.

b. Let I be a 2-sided ideal of R. Show that $M_n(I)$ is a 2-sided ideal of $M_n(R)$.

c. Describe all 2-sided ideals of $M_n(R)$.

5.

(15 points) Let $\bar{A}$ be matrices over a field K with entries indexed by positive integers $\mathbb{N}$ i.e. an element in $\bar{A}$ is a matrix (a_{ij}), $i, j \in \mathbb{N}$. Let A be the subset of $\bar{A}$ consisting of those matrices that have only finitely many non-zero entries.
a. Show that A is a simple algebra over K.
b. Show that A is algebraic over K.

6.

(15 points) Let R be a commutative Noetherian ring and X an indeterminate. Prove that

the power series ring $R[[X]]$ is Noetherian.

7.

(15 points) Let $\mathbb{F}_q$ denote the finite field of $q = p^m$ elements, where p is a prime. Let $\mathrm{GL}_n(\mathbb{F}_q)$ denote the group of non-singular $n \times n$ matrices over $\mathbb{F}_q$.

a. What is the order of $\mathrm{GL}_n(\mathbb{F}_q)$.

b. What is the order of a p-sylow subgroup of $\mathrm{GL}_n(\mathbb{F}_q)$? Find such a p-sylow subgroup.

8.

(15 points) Show that the following polynomials are irreducible over $\mathbb{Q}$ and find their Galois groups.

a. $x^3 - 3x + 1$.

b. $x^3 - 3x - 1$.